PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Part II. Finite Periodic Stationary Gravity Waves in a Perfect Liquid

W. G. Penney and A. T. Price

Phil. Trans. R. Soc. Lond. A 1952 **244**, 254-284
doi: 10.1098/rsta.1952.0004

PART II. FINITE PERIODIC STATIONARY GRAVITY WAVES IN A PERFECT LIQUID

By W. G. PENNEY, F.R.S. and A. T. PRICE

CONTENTS

The possible existence, form and maximum height of strictly periodic finite stationary waves on the surface of a perfect liquid are discussed. A method of successive approximation to the solution of the hydrodynamical equations is formulated, and the solution is carried to the fifth order for the case of two-dimensional waves on a deep liquid. The convergence of the method has not been established, so that the existence of truly periodic stationary waves is not beyond doubt, but the calculations provide strong presumptive evidence for their existence, and for the existence of a finite stable wave of greatest height. The crest of this wave has a right-angled nodal form, in contrast with that of the greatest stable travelling wave for which the nodal angle is 120°. The maximum crest height is $0{\cdot}141\,\lambda$, where λ is the wave-length, and the maximum trough depth is $0{\cdot}078\,\lambda$. This means that the greatest stationary waves are greater than the maximum travelling waves, the ratio being 1·53. The motions of individual particles are studied and it is shown that particles in the surface, particularly those near the anti-nodes have large horizontal motions. For a given wave-length, the period increases with wave height. The wave pressure on a breakwater is examined, and the modification of the calculations to allow for the finite depth of water is considered. Doubly modulated oscillations in a deep rectangular tank are also briefly discussed.

1. Introduction

The General Introduction has explained that the considerations developed in this paper began with some of the breakwater problems of the Mulberry harbours. The emphasis at that time was of course entirely on numerical values of wave pressures acting on floating or fixed breakwaters in finite, even relatively shallow, depths of water; and on the pull on moorings of floating breakwaters. We wish to express our thanks to the Chief of the Royal Naval Scientific Service for permission to use the results which we obtained at that time. However, in attempting to write out our work in a form suitable for publication in the

Philosophical Transactions, the main interest has changed from purely practical numerical values to formal classical hydrodynamics.

Progressive waves on the surface of water have been considered by many writers, and the result obtained by Stokes (1880) that the greatest travelling wave has a pointed crest, enclosing an angle 120°, is well known. By a queer omission, none of the older writers appears to have considered what is the shape of the crest of the greatest possible stationary wave. From an argument somewhat more recondite than that of Stokes, we conclude that the crest of the greatest stable stationary wave has a nodal right-angled form. Possibly there is some other dynamical condition which prevents stable stationary waves reaching such a height, but at any rate, we have not been able to discover such a condition.

A method of successive approximations to the solution of the hydrodynamical equations of stationary waves has been formulated, and in the case of two-dimensional waves in a deep liquid, the solution has been carried to the fifth order. Even for such a high-order calculation, however, the results for the wave of greatest height are not entirely satisfactory, partly because the wave is so big and partly because the mathematical difficulties of attempting to represent a node by a Fourier series are severe. Arguments explained in the following sections provide strong presumptive evidence for the existence of truly periodic stationary waves and of the existence of a wave of greatest height. The algebra could possibly be carried one stage further, but we have not attempted to do this because the results to be obtained did not seem worth the effort required. The nodal crest of the greatest stationary wave cannot be exactly represented by a finite Fourier expansion, and one extra term in the series can hardly affect any of the numerical values found from the fifth-order expansions.

2. Formulation of the mathematical problem

We take rectangular axes with Ox horizontal at the mean level of the water and Oy vertically upwards, the breakwater being in the vertical plane $\mathbf{x} = 0$. The liquid is assumed to be incompressible and moving irrotationally, so that the velocity $(\mathbf{u}, \mathbf{v})$ can be derived from a potential ϕ which satisfies the equation

$$\frac{\partial^2\phi}{\partial\mathbf{x}^2}+\frac{\partial^2\phi}{\partial\mathbf{y}^2}=0, \quad \text{where} \quad \mathbf{u}=-\frac{\partial\phi}{\partial\mathbf{x}}, \quad \mathbf{v}=-\frac{\partial\phi}{\partial\mathbf{y}}. \tag{1}$$

At the breakwater the horizontal velocity $\mathbf{u}$ is always zero so that

$$\frac{\partial\phi}{\partial\mathbf{x}}=0 \quad \text{at } \mathbf{x}=0 \text{ for all } \mathbf{t}. \tag{2}$$

If the water is of uniform depth $\mathbf{d}$, the vertical velocity will be zero at $\mathbf{y} = -\mathbf{d}$, i.e.

$$\frac{\partial\phi}{\partial\mathbf{y}}=0 \quad \text{at } \mathbf{y}=-\mathbf{d} \text{ for all } \mathbf{t}. \tag{3}$$

If the water extends downwards to infinity this condition is replaced by

$$\frac{\partial\phi}{\partial\mathbf{y}}\to 0 \quad \text{as } \mathbf{y}\to-\infty \text{ for all } \mathbf{t}. \tag{3a}$$

Let the free surface of the water at any instant be the surface

$$f(\mathbf{x},\mathbf{y},\mathbf{t})=0. \tag{4}$$

Now a fluid element on this surface must move so that its velocity-component normal to the surface is the same as the normal velocity of the surface itself. Hence the function f must satisfy the condition

$$\frac{\partial f}{\partial \mathbf{t}}+\mathbf{u}\frac{\partial f}{\partial \mathbf{x}}+\mathbf{v}\frac{\partial f}{\partial \mathbf{y}}=0. \tag{5}$$

The pressure at any point in the water is given by Bernoulli's equation

$$\frac{p-p_0}{\rho}=-g\mathbf{y}+\frac{\partial \phi}{\partial \mathbf{t}}-\tfrac{1}{2}(\mathbf{u}^2+\mathbf{v}^2), \tag{6}$$

where ρ is the density and p_0 the atmospheric pressure. At the free surface we have $p=p_0$ and consequently

$$-g\mathbf{y}+\frac{\partial \phi}{\partial \mathbf{t}}-\tfrac{1}{2}(\mathbf{u}^2+\mathbf{v}^2)=0. \tag{7}$$

When ϕ, $\mathbf{u}$ and $\mathbf{v}$ are expressed in terms of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{t}$, this becomes the equation to the free surface, and the left-hand side may be identified with the function $f(\mathbf{x},\mathbf{y},\mathbf{t})$ of equations (4) and (5). Hence the condition (5) is equivalent to the condition

$$\frac{\partial p}{\partial \mathbf{t}}+\mathbf{u}\frac{\partial p}{\partial \mathbf{x}}+\mathbf{v}\frac{\partial p}{\partial \mathbf{y}}=0, \tag{8}$$

used in similar investigations by Stokes and Rayleigh. We find, however, that it is more convenient to apply the condition (5) to an alternative form of the equation to the surface, namely, to equation (11) below.

We now seek those solutions of the above equations which are periodic in $\mathbf{x}$, with wave-length $\lambda=2\pi/k$ say, and examine whether there is one among them which is also periodic in $\mathbf{t}$. By this we mean of course that the wave profile and its rate of change are *exactly* reproduced after a given time interval, say $\mathbf{T}=2\pi/\sigma$. Physical considerations indicate the existence of solutions periodic in $\mathbf{x}$, since these correspond to the finite oscillations of water in a vertically sided trough of width $2\pi/k$, but in general these oscillations will continually change in form, and it is not immediately obvious that there will exist an oscillation which is strictly periodic in the above sense. The problem is thus somewhat similar to that of Stokes's travelling waves, the condition of permanence of form of these waves being now replaced by that of exact reproduction of the wave profile at equal time intervals. The method devised by Levi-Civita (1925) for proving the existence of travelling waves of permanent form is not, however, applicable to the present problem, since it depends essentially on being able to transform the problem to one of steady stream-line flow. We proceed therefore by expressing ϕ as a Fourier series in $\mathbf{x}$ with coefficients which are functions of $\mathbf{t}$, and then approximate to these coefficients as Fourier series in $\mathbf{t}$ by the method of perturbations. The resulting solution is in the form of a double Fourier series in $\mathbf{x}$ and $\mathbf{t}$, with coefficients which are power series in a constant A, where A/π is approximately equal to the ratio wave-height/wave-length. The solution for deep-water waves is carried as far as terms involving A^5, and general formulae are obtained for extending it to any order. For water of finite depth the solution has been taken only to the second order, since it is found that the results differ only slightly from those for infinite depth, if the depth is greater than one-quarter of a wave-length.

The effort required to carry the algebra to A^5 is considerable, but only by going to such high order is it possible to obtain the greatest stationary wave with 2 or 3% accuracy.

3. Stationary waves on deep water

Any velocity potential which is periodic in $\mathbf{x}$ and satisfies (1), (2) and (3*a*) must be expressible in the form

$$\phi = \sum_{n=0}^{\infty} \alpha_n \, e^{nk\mathbf{y}} \cos nk\mathbf{x}, \tag{9}$$

where $\alpha_0, \alpha_1, \ldots$ are functions of $\mathbf{t}$. Substituting in (7) we obtain the equation of the free surface in the form

$$g\mathbf{y} - \sum_{n=0}^{\infty} \dot{\alpha}_n \, e^{nk\mathbf{y}} \cos nk\mathbf{x} + \tfrac{1}{2}k^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn\alpha_m \alpha_n \, e^{(m+n)k\mathbf{y}} \cos (m-n) k\mathbf{x} = 0. \tag{10}$$

The time factors α_n in the expression on the left must be such that the surface condition (5) is satisfied. We shall show that this condition, together with the condition that all the α's are periodic with commensurable periods, determines the α's completely in terms of a constant A depending on the amplitude of the waves. Now an equation of the form (10) may be regarded as an equation to solve for $\mathbf{y}$ in terms of $\mathbf{x}$. Since the coefficients are periodic functions of $\mathbf{x}$, it follows that $\mathbf{y}$ must also be periodic, but it does not follow that $\mathbf{y}$ is necessarily real for all real values of $\mathbf{x}$. To take a trivial example, the equation $\mathbf{y} = 1 - c\, e^{\mathbf{y}} \cos \mathbf{x}$ has a real solution for $\mathbf{y}$ as a function of $\mathbf{x}$ for all real $\mathbf{x}$ only if $C < e^{-2}$. Hence the existence of a continuous free surface must impose some limitation on the α's and consequently on the constant A. This indicates that the waves cannot exceed a certain maximum amplitude. Assuming this condition is satisfied, the solution for $\mathbf{y}$, i.e. the equation of the free surface, will be of the form

$$\mathbf{y} = \tfrac{1}{2}\mathbf{a}_0 + \sum_{n=1}^{\infty} \mathbf{a}_n \cos nk\mathbf{x}, \tag{11}$$

where the $\mathbf{a}$'s are functions of the time. The coefficients $\mathbf{a}_0$ will of course be zero because the axis $\mathbf{y} = 0$ has been taken along the mean level of the water, but it is retained because its presence greatly simplifies and gives a check in the subsequent algebra.

4. Conversion to non-dimensional units

The algebraic labour is very greatly reduced by writing all equations in non-dimensional form. We put

$$x = k\mathbf{x}, \quad y = k\mathbf{y}, \quad a_n = k\mathbf{a}_n = 2\pi \mathbf{a}_n/\lambda, \tag{12}$$

$$t = k^{\frac{1}{2}} g^{\frac{1}{2}} \mathbf{t}, \tag{13}$$

$$\Phi = k^{\frac{3}{2}} g^{-\frac{1}{2}} \phi = \sum_{n=0}^{\infty} \beta_n \, e^{ny} \cos nx, \tag{14}$$

so that

$$\beta_n = k^{\frac{3}{2}} g^{-\frac{1}{2}} \alpha_n. \tag{15}$$

The surface condition (5) can then be expressed in the form

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial y}, \tag{16}$$

and equations (10) and (11) become, respectively,

$$y - \sum_{n=0}^{\infty} \dot{\beta}_n \, e^{ny} \cos nx + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn\beta_m \beta_n \, e^{(m+n)y} \cos (m-n) x = 0 \tag{17}$$

and

$$y = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx. \tag{18}$$

5. Relations between the coefficients in the assumed expansions for Φ and y

Since equation (18) may be regarded as the solution for y in terms of x of equation (17), the result of substituting the value of y from (18) into the left-hand side of (17) must be an expression which is identically zero. Hence, if this expression is expanded as a Fourier series, the coefficients of the separate terms must be zero. This will give a set of relations between the a_n's and β_n's. To do this we require the Fourier expansion of $e^{\lambda y}\cos\mu x$, where λ and μ are integers and y is given by (18). We proceed to find the general terms of this expansion. It is convenient to write (18) in the form

$$y=\frac{1}{2}\sum_{n=-\infty}^{\infty}a_n e^{inx}, \quad \text{where} \quad a_{-n}=a_n. \tag{19}$$

It is then easy to form the product of this expansion with any similar expansion and pick out the coefficient of any exponential term. Thus if

$$z=\frac{1}{2}\sum_{n=-\infty}^{\infty}b_n e^{inx}, \tag{20}$$

we have

$$yz=\frac{1}{4}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_m b_n e^{i(m+n)x}=\frac{1}{4}\sum_{s=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}a_m b_{s-m}\right)e^{isx}. \tag{21}$$

Hence

$$y^2=\frac{1}{4}\sum_{s=-\infty}^{\infty}S_2(s)\,e^{isx}, \quad \text{where} \quad S_2(s)=\sum_{m=-\infty}^{\infty}a_m a_{s-m}. \tag{22}$$

Now let

$$y^N=\frac{1}{2^N}\sum_{s=-\infty}^{\infty}S_N(s)\,e^{isx}; \tag{23}$$

then, using (21), we have

$$y^N=yy^{N-1}=\frac{1}{2^N}\sum_{s=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_m S_{N-1}(s-m)\,e^{isx},$$

which shows that

$$S_N(s)=\sum_{m=-\infty}^{\infty}a_m S_{N-1}(s-m). \tag{24}$$

Continued application of this last result leads to the explicit formula

$$S_N(s)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\ldots a_m a_n a_p \ldots a_{s-m-n-p}\ldots. \tag{25}$$

It will be observed that $S_N(s)=S_N(-s)$ since $a_n=a_{-n}$. Also

$$S_0(0)=1, \quad S_0(s)=0 \text{ for } s\neq 0, \quad \text{and} \quad S_1(s)=a_s \text{ for all } s. \tag{26}$$

We can now write

$$e^{\lambda y}=\sum_{N=0}^{\infty}\frac{\lambda^N}{N!}y^N=\sum_{N=0}^{\infty}\frac{\lambda^N}{2^N N!}\sum_{s=-\infty}^{\infty}S_N(s)\,e^{isx}=\sum_{s=-\infty}^{\infty}E(\lambda,s)\,e^{isx}, \tag{27}$$

where

$$E(\lambda,s)=E(\lambda,-s)=\sum_{N=0}^{\infty}\frac{\lambda^N}{2^N N!}S_N(s). \tag{28}$$

Note that the term corresponding to $N=0$ in the expansion of $E(\lambda,s)$ is zero unless $s=0$, in which case it is unity.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Multiplying the above by $\cos \mu x$ we obtain the required expansion:

$$\begin{aligned} e^{\lambda y} \cos \mu x &= \frac{1}{2} \sum_{s=-\infty}^{\infty} E(\lambda, s) \{e^{i(s+\mu)x} + e^{i(s-\mu)x}\} \\ &= \frac{1}{2} \sum_{s=-\infty}^{\infty} \{E(\lambda, s-\mu) + E(\lambda, s+\mu)\} e^{isx} \\ &= E(\lambda, \mu) + \sum_{s=1}^{\infty} \cos sx \{E(\lambda, s-\mu) + E(\lambda, s+\mu)\}. \end{aligned} \tag{29}$$

We can now substitute the expression (18) for y into equation (17) and use the formula (29). This gives

$$\begin{aligned} \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \dot{\beta}_0 + \sum_{n=1}^{\infty} \dot{\beta}_n \Big[E(n,n) + \sum_{s=1}^{\infty} \cos sx \{E(n, s-n) + E(n, s+n)\} \Big] \\ -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn\beta_m \beta_n \Big[E(m+n, m-n) + \sum_{s=1}^{\infty} \cos sx \{E(m+n, s-m+n) + E(m+n, s+m-n)\} \Big]. \end{aligned}$$

Hence, on equating to zero the coefficients of the separate harmonics, we have

$$\tfrac{1}{2}a_0 = \dot{\beta}_0 + \sum_{n=1}^{\infty} \dot{\beta}_n E(n,n) - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn\beta_m \beta_n E(m+n, m-n) \tag{30}$$

and

$$a_s = \sum_{n=1}^{\infty} \dot{\beta}_n \{E(n, s-n) + E(n, s+n)\} - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn\beta_m \beta_n \{E(m+n, s-m+n) + E(m+n, s+m-n)\}, \tag{31}$$

for all positive integers s.

A second set of relations is obtained by applying the condition (16) to the equation of the surface, which may be taken in either of the forms (17) or (18). Taking the form (18), we write

$$f(x, y, t) = -y + \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos mx, \tag{32}$$

and (16) then gives

$$\begin{aligned} \tfrac{1}{2}\dot{a}_0 + \sum_{m=1}^{\infty} \dot{a}_m \cos mx &= \sum_{m=1}^{\infty} m a_m \sin mx \sum_{n=1}^{\infty} n\beta_n e^{ny} \sin nx - \sum_{n=1}^{\infty} n\beta_n e^{ny} \cos nx \\ &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mna_m \beta_n e^{ny} \{\cos (m-n)x - \cos (m+n)x\} - \sum_{n=1}^{\infty} n\beta_n e^{ny} \cos nx \\ &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mna_m \beta_n \Big[E(n, m-n) - E(n, m+n) \\ &\quad + \sum_{s=1}^{\infty} \cos sx \{E(n, s-m+n) + E(n, s+m-n) - E(n, s-m-n) - E(n, s+m+n)\} \Big] \\ &\quad - \sum_{n=1}^{\infty} n\beta_n \Big[E(n,n) + \sum_{s=1}^{\infty} \cos sx \{E(n, s-n) + E(n, s+n)\} \Big], \end{aligned} \tag{33}$$

on using the formula (29). Since this result is again true for all x, we have, on picking out the separate harmonics,

$$\tfrac{1}{2}\dot{a}_0 = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mna_m \beta_n \{E(n, m-n) - E(n, m+n)\} - \sum_{n=1}^{\infty} n\beta_n E(n,n), \tag{34}$$

and

$$\dot{a}_s = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mna_m\beta_n\{E(n,s-m+n)+E(n,s+m-n)-E(n,s-m-n)-E(n,s+m+n)\}$$
$$-\sum_{n=1}^{\infty} n\beta_n\{E(n,s-n)+E(n,s+n)\}, \tag{35}$$

for all positive integers s.

Now $\dot{a}_0$ is zero, since the water is incompressible and its mean level therefore constant. Hence the right-hand side of (34) is zero; this forms a useful check on later calculations. Since $\dot{a}_0$ is zero, a_0 is constant and becomes zero with our choice of axes. Equation (30) can therefore be regarded as the equation to determine $\dot{\beta}_0$ in terms of the other coefficients; $\dot{\beta}_0$ corresponds to the arbitrary time function in the usual form of Bernoulli's equation, and is required only when calculating the pressure. The form of the free surface is determined by the two infinite sets of equations (31) and (35).

6. The orders of magnitude of the coefficients

If, for the moment, we consider oscillations of very small amplitude, and retain only first-order terms, the equations (31) and (35) reduce to

$$a_s = \dot{\beta}_s \quad \text{and} \quad \dot{a}_s = -s\beta_s, \tag{36}$$

respectively, since for small a_s

$$E(\lambda, 0) \to 1+\tfrac{1}{2}\lambda a_0 \quad \text{and} \quad E(\lambda, s) \to \tfrac{1}{2}\lambda a_s \quad (s \neq 0). \tag{37}$$

Hence we have, approximately,

$$\ddot{a}_s + sa_s = 0, \quad \text{so that} \quad a_s = A_s \cos(t\sqrt{s}+\epsilon_s), \tag{38}$$

and the equation of the free surface becomes

$$y = \sum_{s=1}^{\infty} A_s \cos(t\sqrt{s}+\epsilon_s)\cos sx, \tag{39}$$

where A_s and ϵ_s are arbitrary constants.

The oscillation of the free surface will not in general be strictly periodic because the frequencies of the various components are not in general exact multiples of the fundamental frequency. There are, however, an infinite number of periodic oscillations of small amplitude given by the special cases of (39) in which $A_s = 0$ unless $s = n^2$, where n is an integer. There will, presumably, be a corresponding infinite set of periodic oscillations of finite amplitude, each of which reduces to one of these special cases of (39), when the amplitude is sufficiently reduced. The finite oscillations which are of special interest are those which reduce to a single harmonic term when the amplitude is made small. We shall therefore confine our attention to those oscillations which tend to

$$y = A \sin t \cos x \tag{40}$$

say, as y tends to zero. The coefficients a_1 and β_1 will then be of order A^1 and the remaining coefficients will be of some higher order.

Stokes and Rayleigh, in their treatments of the analogous problem of travelling waves, made assumptions which effectively correspond to taking a_s to be of order A^s. This appears a natural assumption to make, but requires some justification. In the present problem we shall show that, provided only we assume that the α's and β's are expressible in terms of *integral* powers of A, then α_s must be of order A^s and β_s of the same or higher order. We note first that, since

$$E(\lambda, s) = \tfrac{1}{2}\lambda\alpha_s + \tfrac{1}{8}\lambda^2 \sum_{m=-\infty}^{\infty} \alpha_m \alpha_{s-m} + \dots \quad (s \neq 0) \tag{41}$$

and

$$E(\lambda, 0) = 1 + \tfrac{1}{2}\lambda\alpha_0 + \tfrac{1}{8}\lambda^2 \sum_{m=-\infty}^{\infty} \alpha_m \alpha_{-m} + \dots, \tag{42}$$

$E(\lambda, s)$ is at least of the second order unless $s = 1$ or 0, in which cases it is of the first or zero order respectively. Hence to the *second* order of small quantities, the equations (31) and (35) become

$$\begin{aligned} \alpha_1 &= \dot\beta_1 & &\text{and} & \dot\alpha_1 &= -\beta_1, \\ \alpha_2 &= \dot\beta_2 + \tfrac{1}{2}\alpha_1\dot\beta_1 & &\text{and} & \dot\alpha_2 &= -2\beta_2 - \alpha_1\beta_1, \\ \alpha_s &= \dot\beta_s \ (s>2) & &\text{and} & \dot\alpha_s &= -s\beta_s \ (s>2). \end{aligned} \tag{43}$$

It follows that $\ddot\alpha_1 + \alpha_1 = 0$ to the second order, and therefore

$$\alpha_1 = A\sin t, \quad \beta_1 = -A\cos t \quad \text{to the second order.} \tag{44}$$

That is, there are no terms of order A^2 in α_1 or β_1. The second pair of equations in (43) then gives

$$\begin{aligned} \ddot\beta_2 + 2\beta_2 &= -\tfrac{1}{2}\alpha_1\ddot\beta_1 - \tfrac{1}{2}\dot\alpha_1\dot\beta_1 - \alpha_1\beta_1 = (-\tfrac{1}{2}A^2 - \tfrac{1}{2}A^2 + A^2)\sin t\cos t \\ &= 0 \text{ to the second order.} \end{aligned}$$

Hence, for strictly periodic oscillations, β_2 must be *zero to the second order*, since any other solution of the above equation would correspond to an oscillation with a frequency $\sqrt{2}$ times the fundamental frequency. It now follows that

$$\alpha_2 = \tfrac{1}{2}\alpha_1\dot\beta_1 = \tfrac{1}{2}A^2\sin^2 t, \tag{45}$$

so that α_2 is of the second order.

The remaining equations of (43) show that, for $s>2$, $\ddot\alpha_s + s\alpha_s$ and $\ddot\beta_s + s\beta_s$ are both zero to the second order at least. A repetition of the above argument then shows that, unless s is a perfect square, α_s and β_s are zero to the second order. When s is a perfect square, the strict periodicity of the oscillation would not be upset by taking the solution $\alpha_s = A_s\cos(t\sqrt{s} + \epsilon_s)$, where A_s is arbitrary and thus independent of A, but this would be equivalent to taking higher order harmonics in the first-order solution (39), and these we have already excluded. We conclude, therefore, that α_s and β_s for all $s>2$ are of the third or higher order.

We can now prove by induction that, for all s, α_s is of order A^s, and β_s is of the same or higher order. For these statements are true for $s = 1$ and $s = 2$. Suppose now they hold good for all $s<p$ say, and consider the equation (31) for the case $s = p$. From the expressions (41) and (42) for $E(\lambda, s)$, it will be seen that the term $\dot\beta_1 E(1, p-1)$ on the right of (31) (when s is put equal to p) is of order A^p, and no other term is of lower order. Hence $\alpha_p = O(A^p)$.

Now differentiate (31) with respect to t and compare with (35). Remembering that $E(n, p-n) = 1 +$ terms of second and higher orders, when $n = p$, we find that $\ddot\beta_p + p\beta_p$ is equal to an expression of order not less than A^p. Hence, by the same arguments as before, β_p is of order not less than A^p. Thus, if the above statements are true for $s<p$, they are true for $s = p$, and the required result follows by induction.

7. Values of $S_N(s)$ and $E(\lambda, s)$ to the sixth order

In order to find the values of the coefficients α_s and β_s to any particular degree of approximation, say as far as terms in A^p, it is necessary to calculate the function $E(\lambda, s)$, which appears in (31) and (35), as far as A^{p-1}. The functions $S_N(s)$, on which $E(\lambda, s)$ depends, are easily calculated in succession from the recurrence formula (24). Remembering that a_p is of order p, the values of $S_N(s)$ as far as the sixth order have been calculated up to $N=6$ and $s=6$, and are tabulated below. Beyond these values of N and s, $S_N(s)$ is of order higher than the sixth:

$$S_0(0) = 1, \qquad S_0(s) = 0 \quad (s \neq 0),$$
$$S_1(0) = a_0 = 0, \quad S_1(s) = a_s \quad (s \neq 0),$$
$$S_2(0) = 2a_1^2 + 2a_2^2 + 2a_3^2,$$
$$S_2(1) = 2a_1 a_2 + 2a_2 a_3,$$
$$S_2(2) = a_1^2 + 2a_1 a_3 + 2a_2 a_4,$$
$$S_2(3) = 2a_1 a_2 + 2a_1 a_4,$$
$$S_2(4) = a_2^2 + 2a_1 a_3 + 2a_1 a_5,$$
$$S_2(5) = 2a_1 a_4 + 2a_2 a_3,$$
$$S_2(6) = a_3^2 + 2a_1 a_5 + 2a_2 a_4;$$
$$S_3(0) = 6a_1^2 a_2 + 12a_1 a_2 a_3,$$
$$S_3(1) = 3a_1^3 + 6a_1 a_2^2 + 3a_1^2 a_3,$$
$$S_3(2) = 6a_1^2 a_2 + 6a_1 a_2 a_3 + 3a_1^2 a_4 + 3a_2^3,$$
$$S_3(3) = a_1^3 + 3a_1 a_2^2 + 6a_1^2 a_3,$$
$$S_3(4) = 3a_1^2 a_2 + 6a_1 a_2 a_3 + 6a_1^2 a_4,$$
$$S_3(5) = 3a_1 a_2^2 + 3a_1^2 a_3,$$
$$S_3(6) = a_2^3 + 6a_1 a_2 a_3 + 3a_1^2 a_4;$$
$$S_4(0) = 6a_1^4 + 24a_1^2 a_2^2 + 8a_1^3 a_3,$$
$$S_4(1) = 16a_1^3 a_2,$$
$$S_4(2) = 4a_1^4 + 18a_1^2 a_2^2 + 12a_1^3 a_3,$$
$$S_4(3) = 12a_1^3 a_2,$$
$$S_4(4) = a_1^4 + 12a_1^2 a_2^2 + 12a_1^3 a_3,$$
$$S_4(5) = 4a_1^3 a_2,$$
$$S_4(6) = 6a_1^2 a_2^2 + 4a_1^3 a_3;$$
$$S_5(0) = 40a_1^4 a_2, \quad S_6(0) = 20a_1^6,$$
$$S_5(1) = 10a_1^5, \quad S_6(1) = 0,$$
$$S_5(2) = 35a_1^4 a_2, \quad S_6(2) = 15a_1^6,$$
$$S_5(3) = 5a_1^5, \quad S_6(3) = 0,$$
$$S_5(4) = 20a_1^4 a_2, \quad S_6(4) = 6a_1^6,$$
$$S_5(5) = a_1^5, \quad S_6(5) = 0,$$
$$S_5(6) = 5a_1^4 a_2; \quad S_6(6) = a_1^6.$$

The values of $E(\lambda, s)$ up to sixth order can now be written down immediately by substituting the above values of $S_N(s)$ in (28).

8. Equations for a_s and β_s to the fifth order

The above values of $S_N(s)$ are sufficient to carry the calculations of a_s and β_s as far as terms involving A^7. We shall for the present take these calculations only up to A^5. This requires the values of $E(\lambda, s)$ up to the fourth order, and these are easily found to be

$$\left.\begin{aligned} E(\lambda,0) &= 1+\tfrac{1}{4}\lambda^2a_1^2+\tfrac{1}{4}\lambda^2a_2^2+\tfrac{1}{8}\lambda^3a_1^2a_2+\tfrac{1}{64}\lambda^4a_1^4,\\ E(\lambda,1) &= \tfrac{1}{2}\lambda a_1+\tfrac{1}{4}\lambda^2a_1a_2+\tfrac{1}{16}\lambda^3a_1^3,\\ E(\lambda,2) &= \tfrac{1}{2}\lambda a_2+\tfrac{1}{8}\lambda^2a_1^2+\tfrac{1}{4}\lambda^2a_1a_3+\tfrac{1}{8}\lambda^3a_1^2a_2+\tfrac{1}{96}\lambda^4a_1^4,\\ E(\lambda,3) &= \tfrac{1}{2}\lambda a_3+\tfrac{1}{4}\lambda^2a_1a_2+\tfrac{1}{48}\lambda^3a_1^3,\\ E(\lambda,4) &= \tfrac{1}{2}\lambda a_4+\tfrac{1}{8}\lambda^2a_2^2+\tfrac{1}{4}\lambda^2a_1a_3+\tfrac{1}{16}\lambda^3a_1^2a_2+\tfrac{1}{384}\lambda^4a_1^4. \end{aligned}\right\} \tag{46}$$

Substituting these values into equations (30), (31), (34) and (35), we find, after some reduction,

$$a_0 = 2\dot\beta_0+\dot\beta_1(a_1+\tfrac{1}{2}a_1a_2+\tfrac{1}{8}a_1^3)+\dot\beta_2(a_1^2+2a_2)-\beta_1^2-a_1^2\beta_1^2-6a_1\beta_1\beta_2-4\beta_2^2, \tag{47}$$

$$\left.\begin{aligned} a_1 &= \dot\beta_1(1+\tfrac{3}{8}a_1^2+\tfrac{1}{2}a_2+\tfrac{1}{4}a_2^2+\tfrac{1}{4}a_1^2a_2+\tfrac{1}{4}a_1a_3+\tfrac{5}{192}a_1^4)+\dot\beta_2a_1-\beta_1^2(a_1+a_1a_2+\tfrac{1}{2}a_1^3)-2\beta_1\beta_2,\\ a_2 &= \dot\beta_1(\tfrac{1}{2}a_1+\tfrac{1}{2}a_1a_2+\tfrac{1}{2}a_3+\tfrac{1}{12}a_1^3)+\dot\beta_2(1+a_1^2)+\tfrac{3}{2}\dot\beta_3a_1-\beta_1^2(a_2+\tfrac{1}{2}a_1^2)-3\beta_1\beta_2a_1-3\beta_1\beta_3,\\ a_3 &= \dot\beta_1(\tfrac{1}{2}a_2+\tfrac{1}{8}a_1^2+\tfrac{5}{384}a_1^4+\tfrac{1}{8}a_2^2+\tfrac{3}{16}a_1^2a_2+\tfrac{1}{2}a_1a_3+\tfrac{1}{2}a_4)+\dot\beta_2a_1+\dot\beta_3-\beta_1^2(a_3+a_1a_2+\tfrac{1}{6}a_1^3),\\ a_4 &= \dot\beta_1(\tfrac{1}{2}a_3+\tfrac{1}{4}a_1a_2+\tfrac{1}{48}a_1^3)+\dot\beta_2(a_2+\tfrac{1}{2}a_1^2)+\tfrac{3}{2}\dot\beta_3a_1+\dot\beta_4,\\ a_5 &= \dot\beta_1(\tfrac{1}{2}a_4+\tfrac{1}{8}a_2^2+\tfrac{1}{4}a_1a_3+\tfrac{1}{16}a_1^2a_2+\tfrac{1}{384}a_1^4)+\dot\beta_5, \end{aligned}\right\} \tag{48}$$

$$\dot a_0 = 0\beta_1+0\beta_2+0\beta_3+\ldots, \tag{49}$$

$$\left.\begin{aligned} \dot a_1 &= -\beta_1(1+\tfrac{1}{8}a_1^2-\tfrac{1}{2}a_2+\tfrac{1}{192}a_1^4-\tfrac{1}{4}a_1a_3+\tfrac{1}{4}a_2^2)-\beta_2a_1,\\ \dot a_2 &= -\beta_1(a_1+\tfrac{1}{12}a_1^3-a_3)-\beta_2(2+2a_1^2)-3\beta_3a_1,\\ \dot a_3 &= -\beta_1(\tfrac{3}{8}a_1^2+\tfrac{3}{2}a_2+\tfrac{3}{128}a_1^4+\tfrac{3}{16}a_1^2a_2-\tfrac{3}{8}a_2^2-\tfrac{3}{2}a_4)-3\beta_2a_1-3\beta_3,\\ \dot a_4 &= -\beta_1(\tfrac{1}{12}a_1^3+a_1a_2+2a_3)-\beta_2(2a_1^2+4a_2)-6\beta_3a_1-4\beta_4,\\ \dot a_5 &= -\beta_1(\tfrac{5}{384}a_1^4+\tfrac{5}{16}a_1^2a_2+\tfrac{5}{8}a_2^2+\tfrac{5}{4}a_1a_3+\tfrac{5}{2}a_4)-5\beta_5. \end{aligned}\right\} \tag{50}$$

The result (49) that $\dot a_0$ is identically zero is in agreement with previous considerations and forms a useful check on the calculations.

9. Solution of the equations to the third order

We can now, by successive approximations, obtain solutions to the third, fourth and fifth orders of the above simultaneous equations for the a's and β's. We have seen, in § 6, that to the *second order*

$$a_1 = \dot\beta_1, \quad a_2 = \tfrac{1}{2}a_1\dot\beta_1 = \tfrac{1}{2}\dot\beta_1^2, \quad \beta_1 = -A\cos t,$$

and β_2 is of the third or higher order. Substituting these second-order approximations in the appropriate terms on the right of equations (48) and (50), we obtain the following *third-order* approximations:

$$\left.\begin{aligned} a_1 &= \dot\beta_1+\tfrac{5}{8}\dot\beta_1^3-\beta_1^2\dot\beta_1, & \dot a_1 &= -\beta_1+\tfrac{1}{8}\beta_1\dot\beta_1^2,\\ a_2 &= \dot\beta_2+\tfrac{1}{2}\dot\beta_1^2, & \dot a_2 &= -2\beta_2-\beta_1\dot\beta_1,\\ a_3 &= \dot\beta_3+\tfrac{3}{8}\dot\beta_1^3, & \dot a_3 &= -3\beta_3-\tfrac{9}{8}\beta_1\dot\beta_1^2. \end{aligned}\right\} \tag{51}$$

From the first pair of equations, we now find that β_1 must satisfy the equation

$$\ddot{\beta}_1+\tfrac{15}{8}\dot{\beta}_1^2\ddot{\beta}_1-2\dot{\beta}_1^2\beta_1-\ddot{\beta}_1\beta_1^2=-\beta_1+\tfrac{1}{8}\beta_1\dot{\beta}_1^2.$$

In this equation we can substitute the second-order approximation $\ddot{\beta}_1=-\beta_1$ in the third-order terms. We thus get

$$\ddot{\beta}_1+\beta_1=4\dot{\beta}_1^2\beta_1-\beta_1^3. \tag{52}$$

We require now a periodic solution of this equation which, when third-order terms are ignored, reduces to $\beta_1=-A\cos t$. We therefore assume

$$\beta_1=-A\cos\sigma t+\gamma,$$

where γ is of the third order, and σ differs from unity by a quantity of the second order. Substituting in (52) and retaining only third-order terms, we get

$$\sigma^2A\cos\sigma t-A\cos\sigma t+\ddot{\gamma}+\gamma=-4A^3\sin^2\sigma t\cos\sigma t+A^3\cos\sigma t$$
$$=-\tfrac{1}{4}A^3\cos\sigma t+\tfrac{5}{4}A^3\cos 3\sigma t.$$

Equating coefficients of $\cos\sigma t$, we have

$$\sigma^2=1-\tfrac{1}{4}A^2, \tag{53}$$

and therefore $$\ddot{\gamma}+\gamma=\tfrac{5}{4}A^3\cos 3\sigma t,$$

so that $$\gamma=-\tfrac{5}{32}A^3\cos 3\sigma t. \tag{54}$$

Taking now the second pair of equations in (51) we have

$$\ddot{\beta}_2+\dot{\beta}_1\ddot{\beta}_1=-2\beta_2-\beta_1\dot{\beta}_1,$$

and substituting for $\ddot{\beta}_1$ from (53), we find

$$\ddot{\beta}_2+2\beta_2=0 \quad \text{to the third order.}$$

Hence, by the same argument as in § 6, β_2 is zero to the *third* order.

Similarly, from the last pair of equations in (51), we have

$$\ddot{\beta}_3+3\beta_3=0 \quad \text{to the third order,}$$

and therefore β_3 is also zero to the third order.

Hence in the equations (51) we may take β_2 and β_3 zero and

$$\beta_1=-A\cos\sigma t-\tfrac{5}{32}A^3\cos 3\sigma t. \tag{55}$$

10. Solution of the equations to the fourth and fifth orders

We now substitute the approximations (51) into the terms of second or higher order in (48) and (50) and retain terms as far as the fourth order. In doing this, it will be noted that a considerable simplification arises from the fact that β_2, β_3 and β_4 are all of the fourth order at least. We thus obtain

$$\left.\begin{aligned}
a_1&=\dot{\beta}_1+\tfrac{5}{8}\dot{\beta}_1^3-\beta_1^2\dot{\beta}_1, & \dot{a}_1&=-\beta_1+\tfrac{1}{8}\beta_1\dot{\beta}_1^2,\\
a_2&=\dot{\beta}_2+\tfrac{1}{2}\dot{\beta}_1^2+\tfrac{5}{6}\dot{\beta}_1^4-\tfrac{3}{2}\beta_1^2\dot{\beta}_1^2, & \dot{a}_2&=-2\beta_2-\beta_1\dot{\beta}_1-\tfrac{1}{3}\beta_1\dot{\beta}_1^3+\beta_1^3\dot{\beta}_1,\\
a_3&=\dot{\beta}_3+\tfrac{3}{8}\dot{\beta}_1^3, & \dot{a}_3&=-3\beta_3-\tfrac{9}{8}\beta_1\dot{\beta}_1^2,\\
a_4&=\dot{\beta}_4+\tfrac{1}{3}\dot{\beta}_1^4, & \dot{a}_4&=-4\beta_4-\tfrac{4}{3}\beta_1\dot{\beta}_1^3.
\end{aligned}\right\} \tag{56}$$

From the first pair of equations it is found that the equation (52) is true to the fourth order, and consequently β_1 is given by the same expression (55) as before, i.e. β_1 contains no term in A^4.

From the second pair of equations in (56), we obtain after making use of (52),

$$\ddot{\beta}_2+2\beta_2 = 2\beta_1\dot{\beta}_1^3-\beta_1^3\dot{\beta}_1$$

$$= \tfrac{3}{8}A^4\sin 4\sigma t-\tfrac{1}{4}A^4\sin 2\sigma t.$$

Hence

$$\beta_2 = -\tfrac{3}{112}A^4\sin 4\sigma t+\tfrac{1}{8}A^4\sin 2\sigma t. \tag{57}$$

From the last two pairs of equations in (56) it follows that both $\ddot{\beta}_3+3\beta_3$ and $\ddot{\beta}_4+4\beta_4$ are zero to the fourth order. Hence, by arguments similar to those in § 6, β_3 and β_4 are themselves zero to this order.

To carry the solution to the fifth order, we substitute the fourth-order approximations (56) into (48) and (50). This introduces the following additional fifth-order terms which have to be added to the expressions (56) for a_s and $\dot{a}_s$ when s is odd. There are no fifth-order terms in a_s or $\dot{a}_s$ when s is even:

$$\left.\begin{aligned}
[a_1]_5 &= \beta_1^4\dot{\beta}_1-\tfrac{25}{8}\beta_1^2\dot{\beta}_1^3+\tfrac{229}{192}\dot{\beta}_1^5+\tfrac{3}{2}\dot{\beta}_1\dot{\beta}_2-2\beta_1\beta_2,\\
[\dot{a}_1]_5 &= -\tfrac{1}{2}\beta_1^3\dot{\beta}_1^2+\tfrac{55}{192}\beta_1\dot{\beta}_1^4+\tfrac{1}{2}\beta_1\dot{\beta}_2-\dot{\beta}_1\beta_2;\\
[a_3]_5 &= -\tfrac{49}{24}\beta_1^2\dot{\beta}_1^3+\tfrac{409}{384}\dot{\beta}_1^5+\tfrac{3}{2}\dot{\beta}_1\dot{\beta}_2,\\
[\dot{a}_3]_5 &= 3\beta_1^3\dot{\beta}_1^2-\tfrac{159}{128}\beta_1\dot{\beta}_1^4-\tfrac{3}{2}\beta_1\dot{\beta}_2-3\dot{\beta}_1\beta_2;\\
[a_5]_5 &= \dot{\beta}_5+\tfrac{125}{384}\dot{\beta}_1^5,\\
[\dot{a}_5]_5 &= -5\beta_5-\tfrac{805}{384}\beta_1\dot{\beta}_1^4.
\end{aligned}\right\} \tag{56A}$$

From the expressions for a_1 and $\dot{a}_1$ taken to the fifth order, we now find, on eliminating a_1, that β_1 satisfies the differential equation

$$\ddot{\beta}_1+\beta_1 = 4\beta_1\dot{\beta}_1^2-\beta_1^3+2\beta_1\dot{\beta}_1^4-\tfrac{13}{2}\beta_1^3\dot{\beta}_1^2+4\dot{\beta}_1\beta_2+4\beta_1\dot{\beta}_2. \tag{52A}$$

Substituting the fourth-order approximations of β_1 and β_2 into the terms on the right of this equation, and solving the equation, we find that

$$\beta_1 = -A\cos\sigma t-(\tfrac{5}{32}A^3-\tfrac{535}{7168}A^5)\cos 3\sigma t-\tfrac{283}{7168}A^5\cos 5\sigma t, \tag{55A}$$

and

$$\sigma^2 = 1-\tfrac{1}{4}A^2-\tfrac{13}{128}A^4. \tag{53A}$$

Since a_2 and $\dot{a}_2$ contain no fifth-order terms, the differential equation for β_2 is unchanged, and consequently β_2 is given by the same expression (57) as before. Also since a_4 and $\dot{a}_4$ contain no fifth-order terms it follows that $\ddot{\beta}_4+\beta_4$ is zero to the fifth order, so that we can take β_4 as zero also to this order.

From the expressions for a_3 and $\dot{a}_3$ we find

$$\ddot{\beta}_3+3\beta_3 = \tfrac{2}{3}\beta_1\dot{\beta}_1^4-\tfrac{1}{2}\beta_1^3\dot{\beta}_1^2,$$

and hence, on substituting for β_1 and solving the equation,

$$\beta_3 = -\tfrac{1}{96}A^5\cos\sigma t-\tfrac{1}{64}A^5\cos 3\sigma t+\tfrac{7}{2112}A^5\cos 5\sigma t. \tag{58}$$

Similarly, from the expressions for a_5 and $\dot{a}_5$, we find

$$\ddot{\beta}_5+5\beta_5 = -\tfrac{15}{32}\beta_1\dot{\beta}_1^4,$$

and therefore

$$\beta_5 = \tfrac{15}{1024}A^5\cos\sigma t+\tfrac{45}{2048}A^5\cos 3\sigma t-\tfrac{3}{2048}A^5\cos 5\sigma t. \tag{59}$$

11. The wave profile

The coefficients in the expression (18) for the wave profile are obtained as far as terms in A^5 by substituting the above expressions for the β's in (56) and (56A). We find, after some trigonometrical reductions,

$$\left.\begin{aligned} a_1 &= (A+\tfrac{3}{32}A^3-\tfrac{137}{3072}A^5)\sin\sigma t+(\tfrac{1}{16}A^3-\tfrac{11}{5376}A^5)\sin 3\sigma t+\tfrac{163}{21504}A^5\sin 5\sigma t,\\ a_2 &= \tfrac{1}{4}A^2+\tfrac{1}{16}A^4-(\tfrac{1}{4}A^2-\tfrac{25}{192}A^4)\cos 2\sigma t-\tfrac{67}{1344}A^4\cos 4\sigma t,\\ a_3 &= (\tfrac{9}{32}A^3-\tfrac{1}{256}A^5)\sin\sigma t-(\tfrac{3}{32}A^3-\tfrac{2195}{14336}A^5)\sin 3\sigma t-\tfrac{16365}{473088}A^5\sin 5\sigma t,\\ a_4 &= \tfrac{1}{8}A^4-\tfrac{1}{6}A^4\cos 2\sigma t+\tfrac{1}{24}A^4\cos 4\sigma t,\\ a_5 &= \tfrac{145}{768}A^5\sin\sigma t-\tfrac{515}{3072}A^5\sin 3\sigma t+\tfrac{85}{3072}A^5\sin 5\sigma t.\end{aligned}\right\} \quad (60)$$

When $\sigma t = n\pi$, where n is any integer, we have $a_1 = a_3 = a_4 = a_5 = 0$, while $a_2 = \frac{1}{7}A^4$, so that the wave profile is then

$$y = \tfrac{1}{7}A^4\cos 2x. \quad (61)$$

We thus see that there is no instant at which the surface is perfectly flat, the nearest approach to flatness being the modulated surface given by (61). The modulation represented by (61) is, however, very small, and it is unlikely to be noticeable in any real case. Nevertheless, the above result does imply that strictly periodic oscillations of finite amplitude cannot be generated by impulsive pressures applied to the initially flat surface of water at rest.

When $\sigma t = (n+\frac{1}{2})\pi$, where n is an integer, the values of all the β's are zero. Hence u and v are zero everywhere, i.e. the water is momentarily everywhere at rest. At any such instant the wave profile has its maximum amplitude. When n is even, this profile is given by

$$\begin{aligned} y = {} & (A+\tfrac{1}{32}A^3-\tfrac{47}{1344}A^5)\cos x+(\tfrac{1}{2}A^2-\tfrac{79}{672}A^4)\cos 2x\\ & +(\tfrac{3}{8}A^3-\tfrac{12563}{59136}A^5)\cos 3x+\tfrac{1}{3}A^4\cos 4x+\tfrac{295}{768}A^5\cos 5x, \end{aligned} \quad (62)$$

and when n is odd it is given by the same expression except that A must be replaced by $-A$.

It follows that strictly periodic oscillations could be produced, at least in theory, by heaping the water up so that its profile has the form (62) and then letting it move from rest in that position. After a time equal to one-quarter of the period the profile would fall to that given by (61), and after a further equal interval of time it would build up again to the form given by (62) with $-A$ replacing A, i.e. to the same form as the initial profile, except that the crests now occur where the troughs were initially.

It will be seen that the form of the profile at its maximum amplitude is not very different from that of travelling waves of permanent form, which to the fourth order is given by

$$y = a\cos x+(\tfrac{1}{2}a^2+\tfrac{17}{24}a^4)\cos 2x+\tfrac{3}{8}a^3\cos 3x+\tfrac{1}{3}a^4\cos 4x. \quad (63)$$

In both cases the crests become sharper and the troughs flatter as the amplitude is increased.

It is of interest to note that there are no points on the surface of the water which have no vertical movement at all. At the points $x = (n+\frac{1}{2})\pi$, which are true nodes in the case of infinitesimal oscillations, there is an up and down motion given by

$$y = -a_2(t)+a_4(t),$$

so that the water level varies at these points between $-\frac{1}{7}A^4$ and $-\frac{1}{2}A^2+\frac{101}{224}A^4$, the total extent of the movement being $+\frac{1}{2}A^2-\frac{19}{32}A^4$.

We also find that the mean position of the water surface, averaged with respect to time, is not a horizontal plane, but is given by

$$y = (\tfrac{1}{4}A^2 + \tfrac{1}{16}A^4)\cos 2x + \tfrac{1}{8}A^4 \cos 4x, \tag{64}$$

so that there is a rise in the average level at the antinodes ($x = n\pi$) of amount $\frac{1}{4}A^2 + \frac{3}{16}A^4$, and a fall of the average level at the nodes ($x = (n+\frac{1}{2})\pi$) of amount $\frac{1}{4}A^2 - \frac{1}{16}A^4$.

The period of the oscillations is given, in non-dimensional units, by $2\pi/\sigma$, where σ^2 is given by (53A). It follows that the period is *increased*, when compared with that of infinitesimally small oscillations, in the ratio $(1 - \frac{1}{4}A^2 - \frac{13}{128}A^4)^{-\frac{1}{2}}$, i.e. $1 + \frac{1}{8}A^2 + \frac{19}{256}A^4$, approximately.

12. The maximum height of stable two-dimensional periodic stationary waves

We come now to consider the most interesting but the most difficult part of our discussion, namely, the question whether there is a limiting maximum amplitude for two-dimensional periodic stationary waves. In so far as we have passed over the even more difficult mathematical question of the *existence* of finite stationary waves, our treatment will still be logically incomplete. However, if our demonstration on the limiting amplitude survives criticism, we have at least provided *prima facie* evidence that finite stationary waves of a permanent form in a perfect liquid are mathematically possible.

Before beginning our discussion of the stationary waves, it may be helpful and pertinent to make some remarks about the corresponding problems of the existence and maximum amplitude of finite progressive waves. Levi-Civita (1925) has proved that finite waves exist, but in order to establish the convergence of his solution a number of assumptions had to be made which effectively reduce the wave height above mean level to a very small value compared with the wave-length. The greatest value of this ratio which will satisfy all the conditions of Levi-Civita's proof appears to be about 0·005; one of Levi-Civita's inequalities can, however, be replaced by a less drastic one which raises the above ratio to 0·01. Now Stokes (1880) and others, using the method which we have followed and extended in our investigations, showed how to solve the hydrodynamical equations in the form of Fourier series with coefficients which were infinite series in the wave parameter 'a', corresponding with our A, and took the solutions to the fourth power of a. There was nothing in these investigations indicating a maximum value for a; the possibility of a maximum value for a came from a quite independent hydrodynamical point. Stokes, it will be remembered, considered the particles moving near the surface. Making local expansions for the velocity potential and the stream functions near the crest, it is easy to show that the $g\mathbf{y}$ terms in the pressure equation have to balance out with the $\mathbf{q}^2$ terms, and therefore $\mathbf{q}$ was of order $\mathbf{r}^{\frac{1}{2}}$. Consequently, ϕ and ψ were of order $\mathbf{r}^{\frac{3}{2}}$ with which are associated the angular functions $\cos\frac{3}{2}\theta$ and $\sin\frac{3}{2}\theta$. The limiting possible stream line was therefore two straight lines, making a sharp corner at the crest, enclosing an angle $\frac{2}{3}\pi$, i.e. 120°.

The present position on progressive waves is therefore

(1) Stokes, Rayleigh and others solved the hydrodynamical equations as Fourier power series up to any desired order.

(2) Levi-Civita proved that finite but small progressive waves of permanent form do exist.

(3) Stokes showed that, if progressive waves exist having a discontinuity of slope at the crest, the angle there is 120°.

The reasonable attitude may therefore be taken that finite progressive waves exist up to amplitudes consistent with (3), and that the form of these waves can be obtained from the solutions (1) by inserting the limiting condition (3). Michell (1893) discussed the limiting form of progressive waves by a new method, which was extended by Havelock (1918) to waves of smaller height. Havelock showed that, for small amplitudes, Michell's method led to results in good numerical agreement with those obtained by the method of Stokes. The possibility still exists, although it is remote, that there is some other as yet unknown mathematical point (connected with convergence of the series) equivalent to some mechanical condition more stringent than (3) greatly reducing the maximum permissible amplitude to a value more nearly equal to that required in Levi-Civita's proof.

Turning now to finite periodic stationary waves, we have a situation where the investigation corresponding with (1) above has been made in the present paper, but so far we have nothing corresponding with (2) or (3). There seems little likelihood that a proof of the existence of the stationary waves will ever be given, corresponding with Levi-Civita's proof for the progressive waves, because the motion is much more complicated from the mathematical point of view. The form of the surface can be kept constant in time for the progressive waves by using co-ordinates moving with the wave velocity, but no such simplification can be made for the stationary waves. We therefore abandon hope, at least for the present, of proving the existence of stationary waves, and inquire whether there is any hydrodynamical condition, analogous to (3), allowing us to use our Fourier expansions up to the limiting amplitude permitted by this new condition which replaces (3). In other words, we try to do what Stokes and Michell did in obtaining the maximum amplitude of progressive waves, on the assumption that the waves exist.

Our form of the condition limiting the amplitude of the stationary waves may be obtained in several ways, all reducing to the same mathematical condition. We assume for simplicity that the condition applies to the crest at its maximum height. The condition really applies anywhere on the free surface, but as the motion is most extreme at the crest at its greatest height we do not unnecessarily complicate our arguments with a more general discussion, ultimately applying them to the crest.

Our first method of obtaining the limiting condition starts with zero pressure over the free surface. We postulate that the liquid *cannot withstand tension*. Then at the crest (or anywhere else in the surface of the liquid), at all times, the pressure just inside the liquid must be positive or zero and consequently at the crest

$$\frac{\partial p}{\partial \mathbf{y}} \leqslant 0. \tag{65}$$

The y-equation of motion at the point $(\mathbf{x}, \mathbf{y}, \mathbf{t})$ is

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{y}} = -g - \frac{1}{\rho}\frac{\partial p}{\partial \mathbf{y}}.$$

At the instant of greatest height at the crest (and indeed everywhere) $\mathbf{u} = 0$, $\mathbf{v} = 0$ and therefore

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} = -g - \frac{1}{\rho}\frac{\partial p}{\partial \mathbf{y}}. \tag{66}$$

Substituting (65) in (66), we see that

$$-g - \frac{\partial \mathbf{v}}{\partial \mathbf{t}} \leqslant 0. \tag{67}$$

The non-dimensional form of this equation is

$$\frac{\partial v}{\partial t} \geqslant -1. \tag{68}$$

Consequently, the criterion limiting the amplitude of the waves is that the downward acceleration at the crest at its greatest height must not exceed g (or 1 in non-dimensional units).

Now suppose that the pressure over the free surface is not zero, but is positive. The liquid is no longer in tension, but if $\partial p/\partial \mathbf{y}$ is positive at the crest at its greatest height, then from the equation of motion (66), the downward acceleration in the liquid just below the top of the crest is greater than it is at the crest. This we regard as physically untenable.

Our second version of the mechanical condition controlling the maximum amplitude is based on the *stability* of a modulated free surface. Taylor (1950) and Lewis (1950) have shown that if a perfect fluid of lesser density rests on a perfect fluid of greater density and the system is suddenly accelerated downwards with an acceleration greater than g, the interface is unstable. Ripples on the surface grow exponentially, and the exponent is proportional to the inverse square root of the wave-length. Now our analysis shows that waves of wave-lengths submultiples of λ are present in the finite waves. The shorter the submultiple wave-length, the more acute is the instability, once the instability is present. In our case, the lighter fluid is of course of zero density (i.e. vacuum). The criterion of stability, first operative at the crest at its greatest height, is by the theory of Lewis and Taylor

$$g + \frac{\partial \mathbf{v}}{\partial \mathbf{t}} \geqslant 0.$$

This is precisely the same as (67).

The third method of revealing the condition which limits the amplitude of the waves is analytical rather than dynamical.

Our solution of the hydrodynamical motion is based on the assumption that equation (17) for the free surface can be solved for y as a function of x in the form (18), and the condition (16) has then been used to determine the coefficients β_n, appearing in (17), in terms of the parameter A. It is therefore necessary to inquire whether there is actually any finite range of values of A for which the equation (17) has a continuous real solution for y as a function of x, for *all* real values of x and t. If the equation has such a solution, it is necessarily expressible in the form (18).

The maximum permissible value of A for a continuous free surface to exist is given by the maximum value of A defined by the implicit relation

$$p(A, x, y) = 0, \tag{69}$$

where p is the pressure expressed in non-dimensional units. The function $p(A, x, y)$ is in fact the left-hand side of equation (17).

Since A, regarded as a function of real variables x and y, is to be a maximum, we must have

$$\frac{\partial A}{\partial x} = -\frac{\partial p/\partial x}{\partial p/\partial A} = 0, \tag{70}$$

$$\frac{\partial A}{\partial y} = -\frac{\partial p/\partial y}{\partial p/\partial A} = 0. \tag{71}$$

The first of these is apparently satisfied at the crest position because by symmetry $\partial p/\partial x$ is zero, but this result needs closer examination because the crest is a singular point (see later). The second requires $\partial p/\partial y$ to be zero with the same reservations with regard to a singular point. Once again, the equation of motion at the moment of greatest height gives (68) as the criterion of greatest height.

We have now reached the position that if there is some amplitude of the wave such that at the crest at its greatest height the downward acceleration equals g, any attempt further to increase the wave height results in a loss of permanent form. The crests break up and the periodicity is destroyed. There is, however, no *a priori* reason why such a situation should ever arise. The amplitude might indefinitely increase and the period indefinitely increase in such a manner that the downward acceleration never does become equal to or greater than g. Our equations to the fifth order give strong presumptive evidence that there is in fact a limiting wave height.

The numerical value of A in the stationary wave of maximum amplitude may be obtained from (68), using our earlier expressions for Φ. There are two reasons why it is necessary to go to high-order harmonics in order to get reasonable numerical accuracy. The first is that (68) involves a double differentiation of Φ, thus giving an amplification by n^2 of the nth harmonics in the expression for the wave profile (compared with only an amplification by n in the Stokes-Michell progressive wave problem, where the limiting amplitude is obtained from a slope, requiring only a single differentiation). The second unfavourable circumstance is that the maximum value of A is approximately 50 % greater than that for the progressive waves.

At the crests, we have from (14) that

$$\Phi = \Sigma\beta_n \,\mathrm{e}^{ny}.$$

Hence

$$v = -\frac{\partial \Phi}{\partial y}$$

$$= -\Sigma n\beta_n \,\mathrm{e}^{ny}. \tag{72}$$

To apply condition (68), we require $\partial v/\partial t$ at the instant when the crest is highest, i.e. when its velocity v is zero. Denoting the maximum crest height by Y, we have that $\partial v/\partial t$ at the required instant is given by

$$\frac{\partial v}{\partial t} = -\Sigma n\dot{\beta}_n \,\mathrm{e}^{nY}, \tag{73}$$

evaluated at $\sigma t = \frac{1}{2}\pi$.

The values of the first five $\dot{\beta}$'s at $\sigma t = \frac{1}{2}\pi$ follow from § 10 and give to A^5

$$\dot{\beta}_1 = A - \tfrac{19}{32}A^3 + \tfrac{7551}{1792}A^5,$$

$$\dot{\beta}_2 = \quad -\tfrac{5}{14}A^4,$$

$$\dot{\beta}_3 = \quad -\tfrac{7}{132}A^5,$$

$$\dot{\beta}_4 = \quad 0,$$

$$\dot{\beta}_5 = \quad \tfrac{15}{256}A^5.$$

The crest height Y is required only to fourth order and is to fifth order

$$Y = A + \tfrac{1}{2}A^2 + \tfrac{13}{32}A^3 + \tfrac{145}{672}A^4 + \tfrac{2021}{17484}A^5. \tag{74}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Our condition limiting A is $\partial v/\partial t \geqslant -1$. Substituting Y and the $\dot{\beta}$'s in (73) and expanding to the fifth order, we find that

$$A + A^2 + \tfrac{13}{32}A^3 - \tfrac{79}{336}A^4 + \tfrac{331}{7392}A^5 \leqslant 1. \tag{75}$$

The maximum value of A, including terms up to A^5, is given by

$$A = 0{\cdot}592.$$

The convergence in the equation for the maximum value of A and the corresponding peak appears to be good and the accuracy is apparently better than 1 %. However, the point must not be overlooked that such a large value of A means that our replacement of the exponentials in the high-order harmonics by unity is drastic, and the wave profile very near the crest is probably not reliable. In any case, the slope is wrong, since the crest has a node enclosing an angle 90° (see the following section).

Taking $A = 0{\cdot}592$, it will be found that the maximum height of the crest in the greatest stable wave is 0·885 above the level the liquid would have if it were at rest; and the greatest corresponding trough is 0·482 below this same level. Since the wave-length is 2π, we have that the maximum crest height is $0{\cdot}141\lambda$ and the maximum trough depth is $0{\cdot}077\lambda$. The maximum crest to trough distance is therefore $0{\cdot}218\lambda$. The corresponding three figures for the highest stable progressive waves are 0·095, 0·047 and $0{\cdot}143\lambda$. The maximum up and down distance in stationary waves is less than twice the crest to trough distance in progressive waves, the actual ratio being 1·53.

A check on the reliability of the values $A = 0{\cdot}592$, $Y = 0{\cdot}885$ may be had by solving simultaneously for A and y the following pair of equations, retaining the full exponential expressions at $x = 0$, $\sigma t = \frac{1}{2}\pi$

$$p = 0, \quad \partial p/\partial y = 0.$$

One then finds that $A = 0{\cdot}52$, $Y = 0{\cdot}75$, and the wave profile, computed from the isobar $p = 0$, does correctly have a right-angle node at the crest. The value of Y obtained in this way should be appreciably less accurate than that found by the consistent fifth-order approximation, and we therefore expect that the value $Y = 0{\cdot}885$ quoted above is at least accurate to within 2 or 3 %.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

13. The shape of the crest of the greatest stationary wave

An interesting deduction can be made about the shape of the tip of the crest of the greatest stationary wave at the moment of maximum height. Our iterative method, if valid, establishes that the wave motion can be generated precisely by releasing the fluid from rest provided the surface has been shaped to the right form by a constraint which is suddenly removed. Suppose then that the wave of maximum possible height is generated in this way. The dynamical condition which tells us that the wave actually is the one of greatest height is that the initial downward acceleration at the top of the crest, in non-dimensional units, is just unity. This condition is equivalent to $\partial p/\partial y$ zero at the tip.

We shall now show that our solution, when taken to the limit, requires that the angle at the tip of the greatest wave is 90°. Of course, the expressions for the wave profile, when taken to any finite order, since they involve only cosines of nx, necessarily give a wave profile at the crest having a horizontal tangent. Only by taking infinite-order expansions can the true dynamical shape of the profile be represented by a Fourier series. On the other hand, the limiting isobar $p = 0$, the free surface, does correctly have a right-angled crest for the maximum wave if the full exponential expressions are retained. The numerical accuracy which is being achieved to any order is revealed by the agreement between the numerical values of the profile and the isobar $p = 0$ obtained from the pressure equation, retaining the exponentials (see the last paragraph of § 12).

The equation of the free surface of the maximum wave at the instant of greatest elevation may be regarded as given by the implicit relation

$$p(x, y) = 0, \tag{76}$$

where the curves $p(x, y) =$ constant map the isobars in the (x, y) plane. Assuming that $p(x, y)$ is continuous everywhere in the liquid, then for a small displacement $(\delta x, \delta y)$ from the point (x, y) we have

$$\delta p = \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\delta y.$$

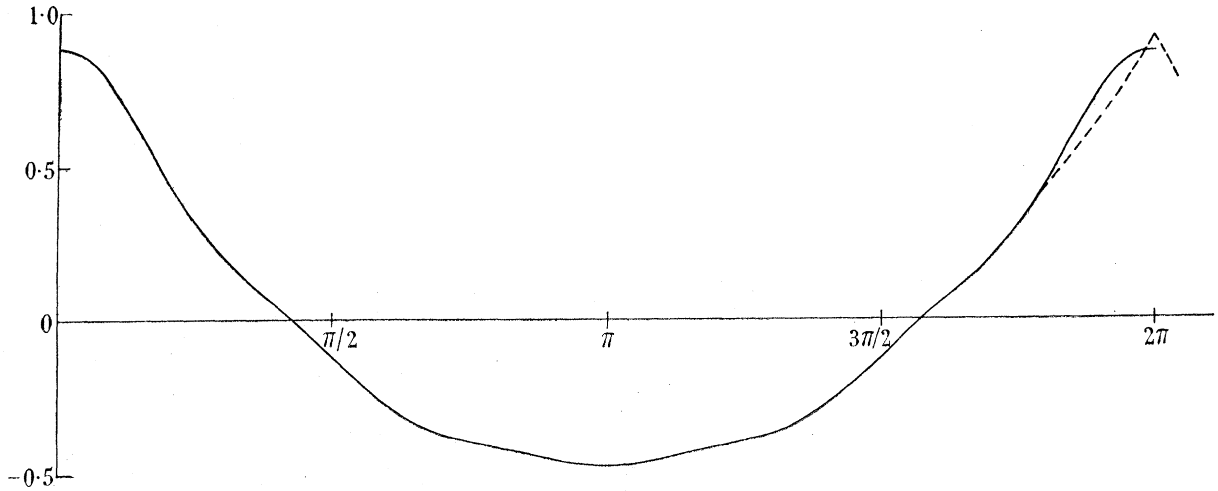


Figure 1. The profile of the greatest stationary wave at its greatest height to fifth order. The scales in the x and y directions are not equal, that in the y direction being $\frac{10}{18}\pi$ (or 1·748) more open than that in the x direction. The fifth-order expressions, like those to any other finite order, give a horizontal tangent at the crest, and we have shown by a dotted line at the right-hand crest our guess at the true profile.

Take the origin at the tip of the crest, and take the displacement $(\delta x, \delta y)$ to be such that the new point also lies in the free surface. Then, for a displacement which never leaves the fluid,

$$0 = \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\delta y. \tag{77}$$

Now, we have already said that at the tip of the crest, $\partial p/\partial y$, is zero. We deduce that $\partial p/\partial x$ is also zero. The tip of the crest, i.e. the origin, is therefore a singular point.

Proceeding to the second order in δx, δy we have, again for a point lying in the free surface near the origin,

$$0 = \frac{\partial^2 p}{\partial x^2}\delta x^2 + 2\frac{\partial^2 p}{\partial x \partial y}\delta x\,\delta y + \frac{\partial^2 p}{\partial y^2}\delta y^2. \tag{78}$$

The y-axis is a line of symmetry. Any differential coefficient involving an odd number of differentiations with respect to x is therefore zero, but there is no corresponding symmetry argument relating to odd differential coefficients in $\partial/\partial y$. We see that $\partial^2 p/\partial x\,\partial y$ is necessarily zero. Furthermore, since p is obtained from a velocity potential by the dimensional equations

$$p/\rho = \dot{\phi} + g\mathbf{y} - \tfrac{1}{2}\mathbf{q}^2,$$

and $\mathbf{q}^2 \equiv 0$ initially, we have that p must satisfy initially

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0. \tag{79}$$

Equation (78) therefore reduces to

$$\frac{\partial^2 p}{\partial x^2}(\delta x^2 - \delta y^2) = 0. \tag{80}$$

Hence, at the origin initially $\qquad \delta y = \pm \delta x.$

The tangents at the tip of the profile therefore meet at right angles. Only if $\partial^2 p/\partial x^2$ is also zero can this conclusion be avoided.

To illustrate the above arguments, let us work out the details using only the lowest harmonic. In non-dimensional units, for this case

$$p = e^y \cos x - y - 1. \tag{81}$$

The equation of the free surface, from which position the motion starts at rest, is the implicit equation corresponding with (76)

$$0 = e^y \cos x - y - 1. \tag{82}$$

At the origin

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= -e^y \sin x = 0, \\ \frac{\partial p}{\partial y} &= e^y \cos x - 1 = 0. \end{aligned}\right\} \tag{83}$$

These relations show that the origin is a singular point.

Again

$$\left.\begin{aligned} \frac{\partial^2 p}{\partial x^2} &= -e^y \cos x = -1, \\ \frac{\partial^2 p}{\partial y^2} &= \quad e^y \cos x = 1, \\ \frac{\partial^2 p}{\partial x\,\partial y} &= -e^y \sin x = 0. \end{aligned}\right\} \tag{84}$$

The second-order differentials thus give

$$\frac{\delta y}{\delta x} = \pm 1. \tag{85}$$

We have proved that if deep liquid at rest, with a surface modulated according to (85), is suddenly released, the acceleration downwards at the tip is just 1, or g in dimensional units.

If the pressure equation (82) for the maximum wave is replaced by a more elaborate pressure equation to any order, obtained by the methods explained in this paper, an easy extension of the above example proves that the angle at the tip of the crest always remains 90°. The value of $\partial^2 p/\partial x^2$ is not zero. We are therefore justified in saying that if our equations are tending to the solution of the stationary wave, then the limiting stationary wave has a right-angled node at the tip of the crest.

A brief digression may now be made to explain a point which puzzled us for some time. Forget that we are discussing stationary waves and consider fluid released from rest with the free surface symmetrically modulated about $x = 0$. Two cases are of interest—one where there is an initial nodal mound pointing upwards, and the other where there is a nodal depression pointing downwards. What are the differential relations at these singular points? The reason why we were puzzled was that the arguments given above at first sight still apply, and therefore in both cases the nodes have to be right-angled.

However, the initial shape is arbitrary, and because this is so, the above argument appears to be fallacious. The dilemma is most easily removed by an appeal to the complex variable, and, from the standard potential theory, the following conclusions are easily reached. In the case of the nodal depression, $\partial p/\partial x$ and $\partial p/\partial y$ become infinite at the node, but their ratio tends to $\pm k$, depending on whether the approach is made from the positive or negative side of the y-axis. The equation of the tangents at the node is $\delta y = \pm k\,\delta x$.

In the case of the nodal elevation, $\partial p/\partial x$, $\partial p/\partial y$, $\partial^2 p/\partial x^2$, $\partial^2 p/\partial x\,\partial y$, $\partial^2 p/\partial y^2$ are all zero. If the angle of the node divides into π exactly m times, then the first differential coefficients of p initially not zero at the node are those of mth order, not odd in $\partial/\partial x$. If the angle of the node divides into π a fractional number $m+j$, where j is between zero and unity, then all differential coefficients of p up to the mth order are initially zero at the node. The differential coefficients of order $(m+1)$ are infinite, but their ratios are such that the $(m+1)$ order differential expressions tend to the solution $\delta y = \pm k\,\delta x$ as the node is approached.

The above digression will be recognized as part of an attempt at the extension of the Poisson-Cauchy theory of the waves generated by an initial disturbance to finite disturbances. The general problem is probably beyond analysis, but numerical solutions could be developed for special cases, and these solutions would have to conform to the differential relations obtained above.

Finally, we make the observation that our conclusion that the crest of the greatest stationary wave at its maximum height is a right-angled node does not depend on the depth of liquid being infinite. Some interesting questions arise on the shape of stationary periodic waves in shallow depths, but we do not attempt the answers here.

14. The particle motion in two-dimensional waves

In this section the paths along which the individual particles move will be considered, the calculations being taken as far as terms in A^4. Consider the particle which is at (x, y) at time $t = 0$. Let it be at $(x+X, y+Y)$ at any subsequent time t, so that its velocity in non-dimensional units is $(\dot{X}, \dot{Y})$. Then

$$\begin{aligned}\dot{X} &= -\frac{\partial \Phi}{\partial x} \quad \text{at } (x+X, y+Y)\\ &= \beta_1\,\mathrm{e}^{y+Y}\sin(x+X) + 2\beta_2\,\mathrm{e}^{2(y+Y)}\sin(2x+2X) \qquad (86)\end{aligned}$$

to the fourth order. Similarly,

$$\dot{Y} = -\beta_1\,\mathrm{e}^{y+Y}\cos(x+X) - 2\beta_2\,\mathrm{e}^{2(y+Y)}\cos(2x+2X). \qquad (87)$$

We note first that, since β_2 is of the fourth order, the equation

$$\frac{dY}{dX} = \frac{\dot{Y}}{\dot{X}} = -\cot(x+X) \tag{88}$$

is true up to the third order. On integrating this equation, we see that the particle must lie on a curve which is given to this order of approximation by

$$Y = -X\cot x + \tfrac{1}{2}X^2 \operatorname{cosec}^2 x - \tfrac{1}{3}X^3 \operatorname{cosec}^2 x \cot x. \tag{89}$$

Hence the particle must oscillate backwards and forwards along an arc of the curve given approximately by (89). This result does not, however, indicate the limits of the motion of the particle, nor its actual position at any particular instant. To determine these it is necessary to integrate the above equations for $\dot{X}$ and $\dot{Y}$ by successive approximations. The first-order solution is easily seen to be

$$X = -A\,e^y \sin x \sin \sigma t, \quad Y = A\,e^y \cos x \sin \sigma t, \tag{90}$$

corresponding to simple harmonic oscillations of amplitude $A\,e^y$ along the straight line $Y = -X\cot x$.

For the second-order solution, we find

$$\dot{X} = -A\cos\sigma t\, e^y (1+Y)(\sin x + X\cos x)$$

$$= -A\,e^y \cos\sigma t(\sin x + X\cos x + Y\sin x) = -A\,e^y\cos\sigma t \sin x$$

on using (90). Hence

$$X = -A\,e^y \sin\sigma t \sin x \quad \text{to the second order.}$$

Similarly $$\dot{Y} = A\,e^y\cos\sigma t(\cos x + A\,e^y \sin\sigma t),$$

so that $$Y = A\,e^y \sin\sigma t\cos x + \tfrac{1}{2}A^2 e^{2y}\sin^2\sigma t.$$

To extend the solution to the third order, we must use the expression (55) for β_1 and take $\sigma = 1-\frac{1}{8}A^2$. Integrating equations (86) and (87) with these values of β_1 and σ and using the above second-order approximations for X and Y, we find

$$\left.\begin{aligned} X &= -\{(A+\tfrac{1}{8}A^3)\sin\sigma t + \tfrac{5}{96}A^3\sin 3\sigma t\}\,e^y \sin x,\\ Y &= \{(A+\tfrac{1}{8}A^3)\sin\sigma t + \tfrac{5}{96}A^3 \sin 3\sigma t\}\,e^y\cos x + \tfrac{1}{2}A^2\sin^2\sigma t\, e^{2y} + \tfrac{1}{3}A^3\sin^3\sigma t\, e^{3y}\cos x. \end{aligned}\right\} \tag{91}$$

It is easily verified that these values of X, Y satisfy the equation (89) to the third order.

For the fourth-order solution we have to introduce β_2 as given by (57) as well as β_1 into equations (86) and (87), together with the above third-order approximations for X and Y. It is sufficient to take $\sigma = 1-\frac{1}{8}A^2$ as before. This leads to

$$\left.\begin{aligned} X &= -\{(A+\tfrac{1}{8}A^3)\sin\sigma t + \tfrac{5}{96}A^3\sin 3\sigma t\}\,e^y\sin x\\ &\quad + \{\tfrac{1}{4}A^4\sin^2\sigma t - \tfrac{3}{112}A^4\sin^2 2\sigma t\}\,e^{2y}\sin 2x,\\ Y &= +\{(A+\tfrac{1}{8}A^3)\sin\sigma t + \tfrac{5}{96}A^3\sin 3\sigma t\}\,e^y\cos x\\ &\quad - \{\tfrac{1}{4}A^4\sin^2\sigma t - \tfrac{3}{112}A^4\sin^2 2\sigma t\}\,e^{2y}\cos 2x\\ &\quad + \{(\tfrac{1}{2}A^2 + \tfrac{9}{32}A^4)\sin^2\sigma t - \tfrac{5}{24}A^4\sin^4\sigma t\}\,e^{2y}\\ &\quad + \tfrac{1}{3}A^3\sin^3\sigma t\, e^{3y}\cos x + \tfrac{1}{12}A^4\sin^4\sigma t\, e^{4y}(2+\cos 2x). \end{aligned}\right\} \tag{92}$$

The particle motion is greatest at the free surface and decreases exponentially with depth below this surface. The paths of the particles which at time $t = 0$ are on the free surface given by (61) are shown in figure 2. Any particle which is on the free surface at time $t = 0$ will remain on the surface for all time. Hence the extremities of the paths of the particles shown in figure 2 should correspond to the position of the free surface when the waves have their maximum amplitude. Using the above equations (92) for the particle motion, it has been verified analytically that any particle which is on the free surface at time $t = 0$ is on the surface as given by (18) and (60), to the fourth order, at time $\sigma t = \frac{1}{2}\pi$.

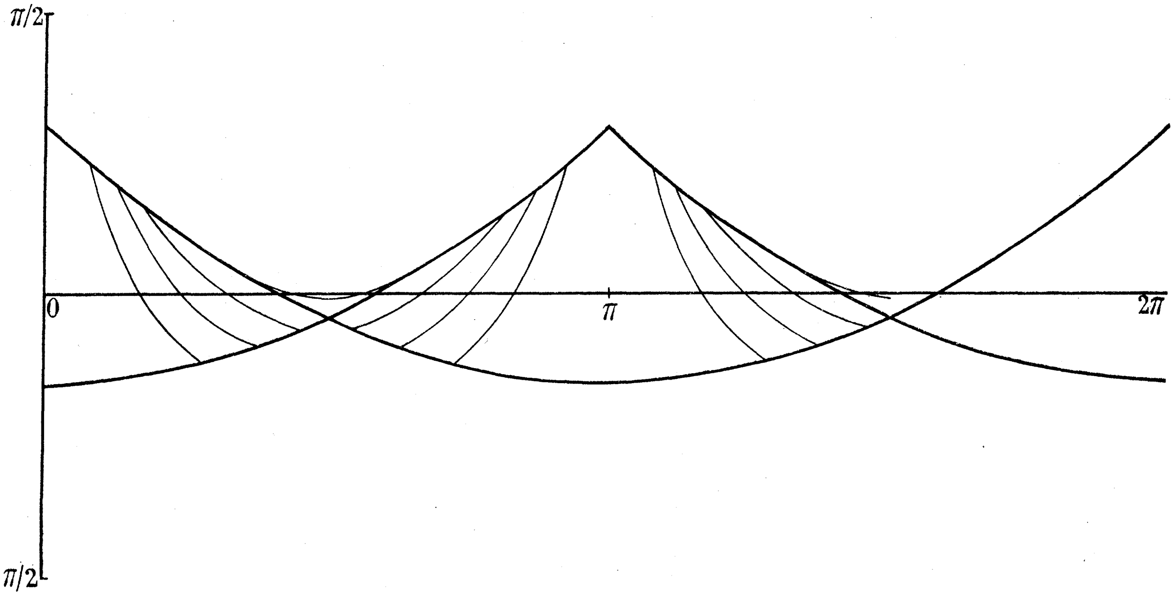


FIGURE 2. Two successive positions of the greatest stationary wave at its position of maximum height. The paths of the particles in the free surface are also shown. Particles in the surface, particularly those near the anti-nodes, have surprisingly large horizontal motions.

15. Oscillations produced when waves of finite height impinge on a vertical breakwater

Suppose that an infinite train of waves of permanent form, of a certain height and wavelength, is travelling in the positive x-direction in a perfect liquid of infinite depth. Then, as follows from the work of Stokes (1880), Rayleigh (1899) and Levi-Civita (1925), the profile of the waves is unique. Suppose now that at some instant a rigid vertical barrier is inserted into the liquid, let us say through the top of one of the wave profiles. Consider the wave pattern on the negative x side of the barrier. A reflected set of waves moves outwards from the barrier, against the oncoming infinite train. It is obvious that in a perfect liquid, a stationary wave pattern will never be set up.

We have at our disposal only the solutions of two essentially steady-state problems, namely, that of travelling waves and that of stationary oscillations. These two steady states cannot co-exist, and we are now concerned with the transition from a state which is represented approximately by one of them to a state represented approximately by the other. In this transition there must enter some physical feature of the phenomena which is not taken into account in the steady-state solutions. We might imagine in the present case, for example, that the fluid motions are subject to slight damping effects, and that, consequently, the disturbance of the incident waves by the breakwater will gradually decrease as we move farther away from it. We thus have the conception of a gradual transition from a state represented approximately by Stokes's travelling waves at great distances from the breakwater to a state represented approximately by stationary oscillations at points close to the breakwater.

Alternatively, we might imagine that the breakwater is only of finite length, say a few wave-lengths long. The reflected waves now give a beam, gradually dispersing, but near the breakwater the pattern is approximately stationary.

The period of the travelling waves is identical with the period of the stationary waves, but the wave-lengths and heights are different. Let us regard as the two independent dynamical variables the wave-length λ_1 of the travelling waves and the coefficient H_1 of the lowest harmonic component $\cos\{(2\pi/\lambda)(\mathbf{x}-c\mathbf{t})\}$, where c is the wave velocity. Let the wave-length of the stationary waves be λ_2 and the coefficient of the lowest harmonic $\cos 2\pi\mathbf{x}/\lambda_2$ be H_2. Then from Stokes's or Rayleigh's results for the travelling waves and from our results for the stationary waves, we have that the period $\mathbf{T}$ of each set is to second order given by

$$\mathbf{T} = \left(\frac{2\pi\lambda_1}{g}\right)^{\frac{1}{2}}\left[1-\frac{2\pi^2H_1^2}{\lambda_1^2}\right]$$

$$= \left(\frac{2\pi\lambda_2}{g}\right)^{\frac{1}{2}}\left[1+\frac{\pi^2H_2^2}{2\lambda_2^2}\right]. \tag{93}$$

From this pair of equations, to second order, it follows that

$$\frac{\lambda_1}{\lambda_2} = 1+\left(\frac{\pi}{\lambda_1}\right)^2(H_2^2+4H_1^2). \tag{94}$$

When the wave heights are small $\quad H_2 = 2H_1.$

Consequently
$$\frac{\lambda_1}{\lambda_2} = 1+8\left(\frac{\pi}{\lambda_1}\right)^2H_1^2. \tag{95}$$

This approximation must still be fairly good when the waves are no longer small, because the two periods are going in opposite directions—the travelling waves go faster with increasing height and therefore for constant wave-length the period decreases with height, while the stationary waves become slower with increasing height. If we put $H_1 = 0{\cdot}05\lambda_1$, corresponding with the longest waves observed at sea, we find

$$\lambda_1/\lambda_2 = 1{\cdot}20.$$

The stationary waves produced by reflexion are roughly 20 % shorter than the incident waves.

16. Two-dimensional oscillations in a deep rectangular tank

The results obtained earlier can be applied directly to the two-dimensional oscillations of a perfect liquid in a deep rectangular tank. The wave-length is such that the tank contains an integral number of half wave-lengths. The depth of liquid in any real tank, of course, will be finite, and might well be small compared with the length of the tank. For this case, we should need to use formulae relating to finite depth rather than those for infinite depth, and the appropriate formulae, at least to second order, are given in § 17.

If λ_n is the wave-length of the nth mode of oscillation, then

$$\lambda_n = 2L/n, \tag{96}$$

where L is the length of the tank.

The wave-length λ_1 of the gravest mode is $2L$, and the oscillation is such that when there is a crest at one end of the tank there is a trough at the other. The next gravest mode has wave-length $\lambda_2 = L$, and the oscillation is such that when there are crests at the two ends there is a trough at the centre, and vice versa.

The heights of the crests and troughs are obtained from (62), and may be put in dimensional form by reverting to (12). The period of oscillation of the nth mode is given by

$$T_n = \frac{(2\pi\lambda_n/g)^{\frac{1}{2}}}{\sigma_n}, \tag{97}$$

where, from (53A), σ_n is given by

$$\sigma^2 = 1 - \tfrac{1}{4}A^2 - \tfrac{13}{128}A^4. \tag{98}$$

The wave profile is fixed by the non-dimensional parameter A, but as shown in § 12, A has an upper limit of 0·592.

The height of the crests above the level which the liquid would have if it were at rest, in the nth mode, is given from (62) or (74) as

$$H_n = \frac{\lambda_n}{2\pi}\{A + \tfrac{1}{2}A^2 + \tfrac{13}{32}A^3 + \tfrac{145}{672}A^4 + 0{\cdot}116A^5\},$$

and the depth of the trough in the nth normal mode is given by

$$D_n = \frac{\lambda_n}{2\pi}\{A - \tfrac{1}{2}A^2 + \tfrac{13}{32}A^3 - \tfrac{145}{672}A^4 + 0{\cdot}116A^5\}.$$

In order to demonstrate the numerical magnitudes of the changes caused by the finite amplitude, consider a tank which has a length $g/2\pi$, say 5·12 ft. The gravest symmetrical oscillation of such a tank in infinitesimal oscillation is 1 sec., and the wave-length of this mode is the length of the tank, i.e. 5·12 ft. Table 1 gives some numerical values for the period, the height of the crests and the depth of the troughs for various values of A.

Table 1

A	T (sec.)	H (ft.)	D (ft.)
0	1·000	0	0
0·2	1·005	0·182	0·149
0·4	1·021	0·418	0·278
0·5	1·034	0·565	0·343
0·592	1·051	0·722	0·394

17. Doubly-modulated oscillations in a deep rectangular tank

An obvious modification of the early sections is to treat purely periodic stationary waves modulated in two directions at right angles. We are now dealing with the truly periodic oscillations of a liquid in a deep rectangular tank, the waves being modulated in both directions.

One might also investigate whether there are truly periodic oscillations in a deep circular tank, the simplest having circular symmetry, while more complicated oscillations are also azimuthally periodic in some integral fraction of 2π. We have not attempted this type of investigation.

The solution is given below up to second order of the doubly-modulated periodic finite waves in a deep rectangular tank.

Taking x- and y-axes horizontal and parallel to the sides of the tank, and the z-axis vertically upwards, we consider those oscillations of the water surface which would approximate to the form

$$\mathbf{z} = a_{11}(\mathbf{t}) \cos l\mathbf{x} \cos m\mathbf{y}$$

when the amplitude is sufficiently reduced.

The general expression for the surface will be of the form

$$\mathbf{z} = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{np}(\mathbf{t}) \cos nl\mathbf{x} \cos pm\mathbf{y}. \tag{99}$$

We find that the coefficients a_{20}, a_{02} and a_{22} are of order a_{11}^2, while the remaining coefficients are of higher order and will therefore be neglected.

The corresponding expression for the velocity potential, as far as the second-order terms, is

$$\phi = \alpha_{11}(\mathbf{t})\, e^{k\mathbf{z}} \cos l\mathbf{x} \cos m\mathbf{y} + \alpha_{20}(\mathbf{t})\, e^{2l\mathbf{z}} \cos 2l\mathbf{x} + \alpha_{02}(\mathbf{t})\, e^{2m\mathbf{z}} \cos 2m\mathbf{y}, \tag{100}$$

where $k^2 = l^2 + m^2$.

On substituting in Bernoulli's equation, we obtain

$$\begin{aligned}\frac{p-p_0}{\rho} &= -g\mathbf{z} + \frac{\partial\phi}{\partial t} - \tfrac{1}{2}(\mathbf{u}^2+\mathbf{v}^2+\mathbf{w}^2) + F(\mathbf{t}) \\ &= -g\mathbf{z} + \dot{\alpha}_{11}\, e^{k\mathbf{z}} \cos l\mathbf{x} \cos m\mathbf{y} + \dot{\alpha}_{20}\, e^{2l\mathbf{z}} \cos 2l\mathbf{x} + \dot{\alpha}_{02}\, e^{2m\mathbf{z}} \cos 2m\mathbf{y} \\ &\quad - \tfrac{1}{2}\alpha_{11}^2\, e^{2k\mathbf{z}}\,(m^2 \cos^2 l\mathbf{x} + l^2 \cos^2 m\mathbf{y}) + F(\mathbf{t}).\end{aligned} \tag{101}$$

At the surface p is equal to p_0; hence substituting from (99) in (101) and retaining terms up to the second order only, we get

$$\begin{aligned}&-g(a_{11} \cos l\mathbf{x} \cos m\mathbf{y} + a_{20} \cos 2l\mathbf{x} + a_{02} \cos 2m\mathbf{y} + a_{22} \cos 2l\mathbf{x} \cos 2m\mathbf{y}) \\ &\quad + \dot{\alpha}_{11}\{\cos l\mathbf{x} \cos m\mathbf{y} + \tfrac{1}{4}ka_{11}(1 + \cos 2l\mathbf{x} + \cos 2m\mathbf{y} + \cos 2l\mathbf{x} \cos 2m\mathbf{y})\} \\ &\qquad + \dot{\alpha}_{20} \cos 2l\mathbf{x} + \dot{\alpha}_{02} \cos 2m\mathbf{y} - \tfrac{1}{4}\alpha_{11}^2(k^2 + l^2 \cos 2m\mathbf{y} + m^2 \cos 2l\mathbf{x}) + F(\mathbf{t}) = 0,\end{aligned}$$

for all $\mathbf{x}$ and $\mathbf{y}$. Equating to zero the coefficients of the separate trigonometrical terms gives

$$F(t) = \tfrac{1}{4}k^2\alpha_{11}^2 - \tfrac{1}{4}ka_{11}\dot{\alpha}_{11}, \tag{102}$$

$$-ga_{11} + \dot{\alpha}_{11} = 0, \tag{103}$$

$$-ga_{20} + \dot{\alpha}_{20} - \tfrac{1}{4}m^2\alpha_{11}^2 + \tfrac{1}{4}ka_{11}\dot{\alpha}_{11} = 0, \tag{104}$$

$$-ga_{02} + \dot{\alpha}_{02} - \tfrac{1}{4}l^2\alpha_{11}^2 + \tfrac{1}{4}ka_{11}\dot{\alpha}_{11} = 0, \tag{105}$$

$$-ga_{22} + \tfrac{1}{4}ka_{11}\dot{\alpha}_{11} = 0. \tag{106}$$

We also have at the surface the condition

$$\frac{\partial p}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial p}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial p}{\partial \mathbf{y}} + \mathbf{w}\frac{\partial p}{\partial \mathbf{z}} = 0,$$

leading to the identity

$$\ddot{\alpha}_{11}\cos l\mathbf{x}\cos m\mathbf{y}+\tfrac{1}{4}ka_{11}\ddot{\alpha}_{11}(1+\cos 2l\mathbf{x})(1+\cos 2m\mathbf{y})+\ddot{\alpha}_{20}\cos 2l\mathbf{x}+\ddot{\alpha}_{02}\cos 2m\mathbf{y}$$
$$-\tfrac{1}{2}\alpha_{11}\dot{\alpha}_{11}(k^2+m^2\cos 2l\mathbf{x}+l^2\cos 2m\mathbf{y})+\dot{F}(\mathbf{t})$$
$$-\tfrac{1}{4}\dot{\alpha}_{11}\dot{\alpha}_{11}\{l^2(1-\cos 2l\mathbf{x})(1+\cos 2m\mathbf{y})+m^2(1+\cos 2l\mathbf{x})(1-\cos 2m\mathbf{y})$$
$$+k^2(1+\cos 2l\mathbf{x})(1+\cos 2m\mathbf{y})\}$$
$$+\alpha_{11}kg\cos l\mathbf{x}\cos m\mathbf{y}+\tfrac{1}{4}\alpha_{11}^2k^2g(1+\cos 2l\mathbf{x})(1+\cos 2m\mathbf{y})$$
$$+2lg\alpha_{20}\cos 2l\mathbf{x}+2mg\alpha_{02}\cos 2m\mathbf{y}=0,$$

where terms up to the second order only have been retained. The five equations obtained by equating to zero the coefficients of the separate terms are all satisfied if

$$\ddot{\alpha}_{11}+kg\alpha_{11}=0, \tag{107}$$

$$\ddot{\alpha}_{20}+2lg\alpha_{20}=m^2\alpha_{11}\dot{\alpha}_{11}, \tag{108}$$

$$\ddot{\alpha}_{02}+2mg\alpha_{02}=l^2\alpha_{11}\dot{\alpha}_{11} \tag{109}$$

and

$$F(\mathbf{t})=k^2\alpha_{11}\dot{\alpha}_{11}. \tag{110}$$

From (107) we have, with a suitable choice of time origin,

$$\alpha_{11}=A\cos\sigma\mathbf{t}, \tag{111}$$

where

$$\sigma^2=kg. \tag{112}$$

Then from (108),

$$\alpha_{20}=-\frac{1}{D^2+2lg}m^2A^2\sigma\sin\sigma\mathbf{t}\cos\sigma\mathbf{t}=\frac{m^2\sigma A^2}{4g(2k-l)}\sin 2\sigma\mathbf{t}, \tag{113}$$

and similarly from (109),

$$\alpha_{02}=\frac{l^2\sigma A^2}{4g(2k-m)}\sin 2\sigma\mathbf{t}. \tag{114}$$

This determines the expression for the velocity potential ϕ as far as terms of the second order. For the coefficients in the equation to the surface, we have from (103) to (106),

$$a_{11}=-\frac{A\sigma}{g}\sin\sigma\mathbf{t}, \tag{115}$$

$$a_{20}=\frac{A^2l^2}{8g}\left\{1+\frac{2k^2-2kl-l^2}{2kl-l^2}\cos 2\sigma\mathbf{t}\right\}, \tag{116}$$

$$a_{02}=\frac{A^2m^2}{8g}\left\{1+\frac{2k^2-2km-m^2}{2km-m^2}\cos 2\sigma\mathbf{t}\right\}, \tag{117}$$

$$a_{22}=\frac{A^2k^2}{8g}\{1-\cos 2\sigma\mathbf{t}\}. \tag{118}$$

Also from (102) we find

$$F(\mathbf{t})=\tfrac{1}{4}k^2A^2\cos 2\sigma\mathbf{t}. \tag{119}$$

It will be noted that this result is consistent with the expression (110) for $\dot{F}(\mathbf{t})$.

At time $\mathbf{t}=0$ we have $a_{11}=0$ and $a_{22}=0$, but

$$a_{20}=\frac{A^2lm^2}{4g(2k-l)}\quad\text{and}\quad a_{02}=\frac{A^2l^2m}{4g(2k-m)}. \tag{120}$$

Hence there is no instant at which the water surface is perfectly flat, the nearest approach to flatness being the modulated surface

$$z = \frac{A^2 lm}{4g}\left\{\frac{m}{2k-l}\cos 2l\mathbf{x} + \frac{l}{2k-m}\cos 2m\mathbf{y}\right\}. \tag{121}$$

This surface is of the second order, except when l or m is zero, i.e. except when the surface is modulated in one direction only. If the latter case it has been seen in § 11, equation (61), that the corresponding surface is of the fourth order.

18. Stationary waves on water of finite depth

When the water extends downwards to a uniform depth d below the level of the free surface, the velocity potential may be expressed in the form

$$\phi = \sum_{n=0}^{\infty} \alpha_n(\mathbf{t}) \cosh\{nk(\mathbf{y}+\mathbf{d})\} \operatorname{cosech} nk\mathbf{d} \cos nk\mathbf{x}. \tag{122}$$

This satisfies the condition that the vertical component $(-\partial\phi/\partial\mathbf{y})$ of the velocity is zero at $\mathbf{y} = -\mathbf{d}$. It reduces to the expression (9) when $\mathbf{d}$ becomes infinite.

Substituting in (7), we have the equation to the free surface in the form

$$-g\mathbf{y} + \sum_{n=0}^{\infty} \dot{\alpha}_n(\mathbf{t}) \operatorname{cosech} nk\mathbf{d} \cosh\{nk(\mathbf{y}+\mathbf{d})\} \cos nk\mathbf{x}$$
$$-\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnk^2\alpha_m\alpha_n \operatorname{cosech} mk\mathbf{d} \operatorname{cosech} nk\mathbf{d}[\cosh\{k(m+n)(\mathbf{y}+\mathbf{d})\}\cos\{(m-n)k\mathbf{x}\}$$
$$-\cosh\{k(m-n)(\mathbf{y}+\mathbf{d})\}\cos\{(m+n)k\mathbf{x}\}] = 0. \tag{123}$$

We assume, as before, that the equation to the free surface can also be reduced to the form

$$f(\mathbf{x},\mathbf{y},\mathbf{t}) \equiv -\mathbf{y} + \tfrac{1}{2}a_0(\mathbf{t}) + \sum_{n=1}^{\infty} a_n(\mathbf{t}) \cos nk\mathbf{x} = 0. \tag{124}$$

Then by substituting the value of $\mathbf{y}$ given by (124) into (123) we obtain an identity in $\mathbf{x}$, from which we can derive a set of relations between the a's and the α's, analogous to the previous equations (30) and (31). We shall here consider these relations only as far as terms of the second order. We find to this order

$$-\tfrac{1}{2}ga_0 + \dot{\alpha}_0 + \tfrac{1}{2}ka_1\dot{\alpha}_1 - \tfrac{1}{2}k^2\alpha_1 \operatorname{cosech}^2 k\mathbf{d} \cosh 2k\mathbf{d} = 0, \tag{125}$$

$$-ga_1 + \dot{\alpha}_1 \coth k\mathbf{d} = 0, \tag{126}$$

$$-ga_2 + \tfrac{1}{2}ka_1\dot{\alpha}_1 + \dot{\alpha}_2 \coth 2k\mathbf{d} + \tfrac{1}{2}k^2\alpha_1^2 \operatorname{cosech}^2 k\mathbf{d} = 0. \tag{127}$$

The surface condition (5) leads to another set of relations, which to the second order reduce to

$$\dot{a}_0 \equiv 0, \tag{128}$$

$$\dot{a}_1 + k\alpha_1 = 0, \tag{129}$$

$$\dot{a}_2 + 2k\alpha_2 + k^2 a_1\alpha_1 \coth k\mathbf{d} = 0. \tag{130}$$

From (126) and (129) we have

$$\ddot{a}_1 = -k\dot{\alpha}_1 = -(kg \tanh k\mathbf{d})\, a_1.$$

Hence we can take $$a_1 = B\sin\sigma\mathbf{t}, \tag{131}$$

where $$\sigma^2 = kg\tanh k\mathbf{d}. \tag{132}$$

This shows that the period increases as the depth is decreased. For a depth equal to one-half of a wave-length the increase compared with that for infinite depth is, however, less than 0·2 %; for a depth of one-quarter of a wave-length it is a little less than 5 %.

To obtain the value of a_2 we have, on eliminating α_2 between (127) and (130) and substituting for a_1 and α_1,

$$\ddot{a}_2\coth 2k\mathbf{d}+2kga_2 = k\sigma^2B^2\{\tfrac{1}{2}\coth^2 k\mathbf{d}+(\tfrac{1}{2}+\operatorname{cosech}^2 k\mathbf{d})\cos 2\sigma\mathbf{t}\},$$

and solving this equation

$$a_2 = \tfrac{1}{4}kB^2\coth k\mathbf{d}-\tfrac{1}{4}kB^2(\coth k\mathbf{d}+2\coth k\mathbf{d}\operatorname{cosech}^2 k\mathbf{d})\cos 2\sigma\mathbf{t}. \tag{133}$$

As the depth $\mathbf{d}$ of the water is increased this result approximates to the value already found for a_2 when the depth is infinite. For a depth equal to one-half of a wave-length ($k\mathbf{d}=\pi$), we have $\coth k\mathbf{d} = 1{\cdot}0037$ and $\operatorname{cosech}^2 k\mathbf{d} = 0{\cdot}008$, so that the result would differ negligibly from that for infinite depth.

For very small values of $\mathbf{d}$, the factors $\coth k\mathbf{d}$ and $\operatorname{cosech} k\mathbf{d}$ will make a_2 large compared with a_1 unless B is sufficiently small. Similar hyperbolic factors will occur in the coefficients of the higher harmonics. It is evident therefore that the Fourier expansion (124) for the free surface can be convergent (if at all) only for sufficiently small values of B whose upper bound is dependent on the depth $\mathbf{d}$. Hence if periodic stationary waves on water of finite depth exist, their maximum amplitude will depend on the depth and will tend to zero as the depth is decreased.

19. Wave pressure on a breakwater (deep water; normal incidence)

An interesting application of the theory developed in the preceding sections is to compute the pressure on a breakwater due to wave action. The most interesting examples are those where there is a crest or a trough at the breakwater.

The general equation for the pressure in the liquid, in dimensional form, is given by (6), but in this equation we may at once write p_0 zero, since we are not interested in the atmospheric pressure, and we may also write $\mathbf{u}^2+\mathbf{v}^2=0$, since we are at present only concerned with the pressure at the highest and lowest positions of the surface.

Thus, the pressure on the breakwater is given by

$$\begin{aligned} p &= -\rho g\mathbf{y}+\rho\frac{\partial\phi}{\partial\mathbf{t}} \\ &= -\rho g\mathbf{y}+\frac{\lambda\rho g}{2\pi}\frac{\partial\Phi}{\partial t}, \end{aligned} \tag{134}$$

the second equation involving the non-dimensional velocity potential and time.

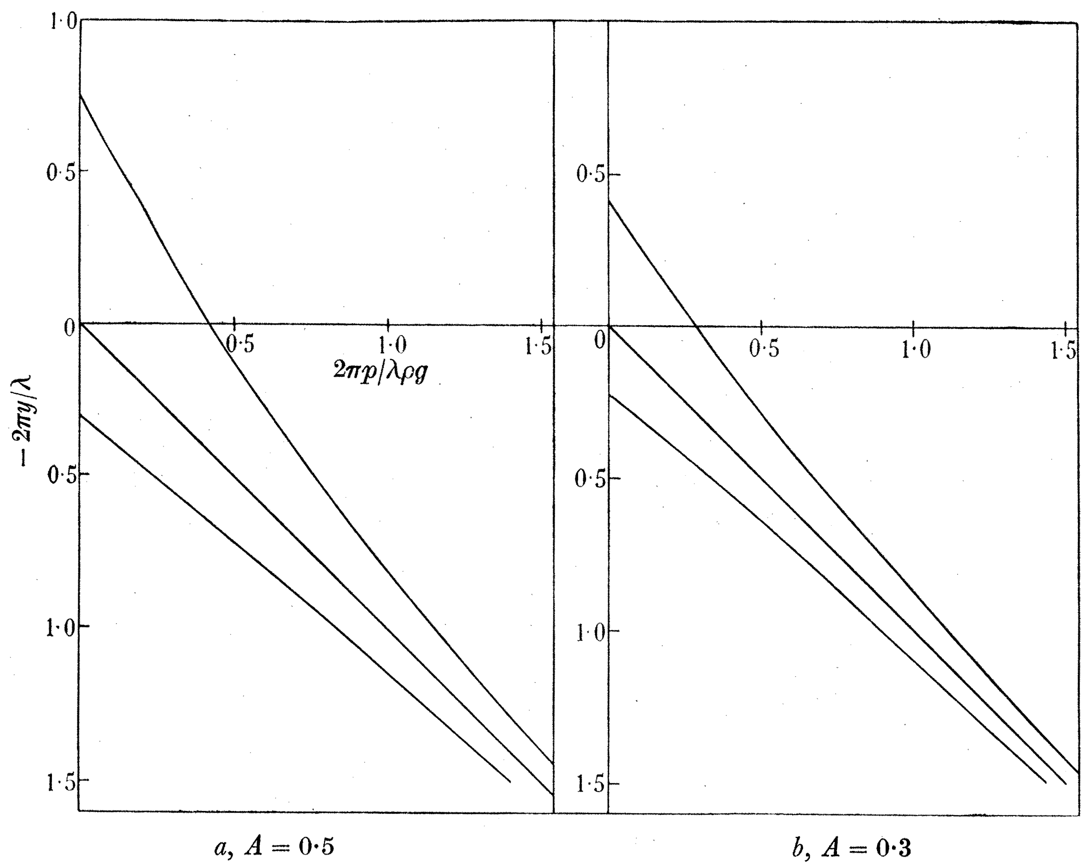
Now $$\frac{\partial\Phi}{\partial t} = \Sigma\dot{\beta}_n e^{ny}\cos nx,$$

and the β_n^s can be obtained at the moment of extreme surface modulation from § 10. To obtain the pressure at a crest, we write $x = 0$, so that all $\cos nx$ are unity; and to obtain the pressure at a trough, we write $x = \pi$, so that $\cos x$, $\cos 3x$, etc., are -1, and $\cos 2x$, $\cos 4x$, etc., are $+1$.

We find that the hydrostatic pressures on the breakwater at a crest (upper signs) and the hydrostatic pressure at a trough (lower signs) are given by

$$p = \frac{\lambda \rho g}{2\pi}\{-y \pm (A - \tfrac{19}{32}A^3 + \tfrac{755}{1792}A^5)\,e^y - \tfrac{5}{14}A^4\,e^{2y} \mp (\tfrac{7}{132}A^5)\,e^{3y} \pm (\tfrac{15}{256}A^5)\,e^{5y}\}, \qquad (135)$$

where the y in { } is non-dimensional, i.e. the dimensional y divided by $\lambda/2\pi$.



FIGURES 3 a, b. The pressure distribution in the liquid, in units $\lambda\rho g/2\pi$, vertically below the crest and the trough at the moment of greatest surface modulation for the two cases $A = 0·5$ and $A = 0·3$. The vertical unit of distance is also given in non-dimensional form, i.e. the dimensional value of y must be multiplied by $2\pi/\lambda$ to get the numerical value of the ordinate. The straight lines through the origin show the hydrostatic pressure which would exist if the wave height were zero. The pressure curves for all times are asymptotic to these straight lines for large negative y because at great depths the effects of wave motion disappear.

Figure 3 a plots the pressure in units of $\lambda\rho g/2\pi$ in terms of the non-dimensional y in the case $A = 0·5$. The upper curve relates to the crest and the lower curve to the trough. For large negative y, both curves asymptotically approach $-y$. For $y \geqslant 0$, of course, to be consistent, we should expand the exponentials, regard y as of order A, and take only the terms in p up to fifth order. The agreement with p as computed from the expression (135) is, however, quite satisfactory, except very close to the tip. According to the fifth-order expansions with $A = 0{\cdot}5$, the crest height is 0·693, whereas equation (135) makes the pressure vanish at $y = 0{\cdot}74$.

Figure 3*b* is similar to 3*a*, but relates to the value $A = 0{\cdot}3$. There is no difficulty this time at the crest because the terms with the larger exponentials make much smaller contributions.

References

Havelock, T. H. 1918 *Proc. Roy. Soc.* A, **95**, 38.
Lewis, D. J. 1950 *Proc. Roy. Soc.* A, **202**, 81.
Levi-Civita, T. 1925 *Math. Ann.* **93**, 264.
Michell, J. H. 1893 *Phil. Mag.* (5), **36**, 430.
Rayleigh, Lord 1899–1920 *Collected papers*, **1**, 197 and 314. Cambridge University Press.
Stokes, G. G. 1880 *Collected papers*, **5**, 62. Cambridge University Press.
Taylor, G. I. 1950 *Proc. Roy. Soc.* A, **201**, 192.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES